

Parametric Stokes Phenomenon for the second Painlevé equation with a large parameter

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Abstract

The second Painlevé equation with a large parameter (P_{II}) is analyzed by using the exact WKB analysis. The purpose of this study is to investigate the problem of the degeneration of P -Stokes geometry of (P_{II}), which relates to a kind of Stokes phenomena for asymptotic (formal) solutions of (P_{II}). We call this Stokes phenomenon a “parametric Stokes phenomenon”. We formulate the connection formula for this Stokes phenomenon, and confirm it in two ways: the first one is by computing the “Voros coefficient” of (P_{II}), and the second one is by using the isomonodromic deformation theory. Our main claim is that the connection formulas derived by these two completely different methods coincide.

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1 Introduction and main results

The theory of the exact WKB analysis of Painlevé equations with a large parameter has been established in the series of papers [KT2], [AKT1] and [KT3]. The main result of these papers is that any 2-parameter (formal) solution of the J -th Painlevé equation ($J = \text{II}, \dots, \text{VI}$), which has been constructed through multiple-scale analysis in [AKT1], can be transformed formally to a 2-parameter solution of the first Painlevé equation (P_1) near a simple P -turning point. (In this paper we call turning points (resp. Stokes curves) of Painlevé equations, the definition of which is given in [KT2], P -turning points (resp. P -Stokes curves). These terminologies have been already used in the exact WKB analysis of higher order Painlevé equations: cf. [KT5].) Moreover, the connection formulas for 2-parameter solutions of (P_1) on P -Stokes curves were also discussed in [T1] by using the isomonodromic deformation method, that is, by analyzing linear differential equations associated with (P_1) through the exact WKB analysis.

The analysis presented in the above papers is concerned with the local theory near a simple P -turning point. But, for (P_J) ($J = \text{II}, \dots, \text{VI}$), two or more P -turning points appear in general and “a degeneration of P -Stokes geometry” sometimes occurs; that is, a P -Stokes curve may connect two P -turning points when parameters contained in (P_J) take some special values. For example, the second Painlevé equation with a large parameter η

$$(P_{\text{II}}) : \frac{d^2\lambda}{dt^2} = \eta^2(2\lambda^3 + t\lambda + c)$$

has three P -turning points if $c \neq 0$, and Figure 1.1 ~ 1.3 describe the P -Stokes curves of (P_{II}) near $\arg c = \frac{\pi}{2}$. (See Section 2.2 for the definitions of P -turning points and P -

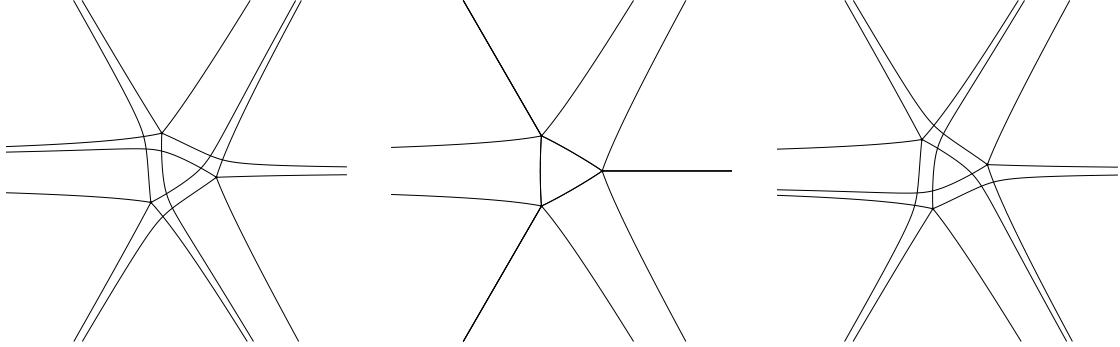


Figure 1.1: P -Stokes curves when $\arg c = \frac{\pi}{2} - \varepsilon$.

Figure 1.2: P -Stokes curves when $\arg c = \frac{\pi}{2}$.

Figure 1.3: P -Stokes curves when $\arg c = \frac{\pi}{2} + \varepsilon$.

Stokes curves of (P_{II} .) As is clear from Figure 1.1 ~ 1.3, a degeneration of P -Stokes geometry is observed when $\arg c = \frac{\pi}{2}$. This degeneration suggests that a kind of Stokes phenomena occurs when c varies near $\arg c = \frac{\pi}{2}$, that is, the correspondence between asymptotic solutions and true solutions of (P_{II}) changes discontinuously before and after the degeneration. We call this phenomenon “a parametric Stokes phenomenon” because this Stokes phenomenon (or the degeneration of P -Stokes geometry) occurs when the parameter c contained in (P_{II}) varies. The purpose of this paper is to investigate the parametric Stokes phenomenon for (P_{II}) in an explicit manner.

In this paper we mainly discuss the parametric Stokes phenomenon for the following 1-parameter family of transseries solutions (1-parameter solutions) of (P_{II}):

$$\lambda(t, c, \eta; \alpha) = \lambda^{(0)}(t, c, \eta) + \alpha \eta^{-\frac{1}{2}} \lambda^{(1)}(t, c, \eta) e^{\eta \phi_{\text{II}}} + (\alpha \eta^{-\frac{1}{2}})^2 \lambda^{(2)}(t, c, \eta) e^{2\eta \phi_{\text{II}}} + \dots \quad (1.1)$$

Here α is a free parameter, $\lambda^{(k)}(t, c, \eta) = \lambda_0^{(k)}(t, c) + \eta^{-1} \lambda_1^{(k)}(t, c) + \eta^{-2} \lambda_2^{(k)}(t, c) + \dots$ ($k \geq 0$) are formal power series of η^{-1} and $\phi_{\text{II}} = \phi_{\text{II}}(t, c)$ is some function. (Note that $\lambda^{(0)}(t, c, \eta)$

itself is a formal power series solution of (P_{II}) , called 0-parameter solution.) In Section 3.1 we introduce two normalizations of 1-parameter solutions; let $\lambda_{\infty}(t, c, \eta; \alpha)$ (resp. $\lambda_{\tau}(t, c, \eta; \alpha)$) be a 1-parameter solution which is normalized at $t = \infty$ (resp. at some P -turning point $t = \tau$). (A construction of 1-parameter solutions are briefly reviewed in Section 2.1.) Our aim is to explicitly determine the connection formulas for the parametric Stokes phenomenon occurring to 1-parameter solutions $\lambda_{\infty}(t, c, \eta; \alpha)$ and $\lambda_{\tau}(t, c, \eta; \alpha)$ when $\arg c = \frac{\pi}{2}$.

The motivation of this paper comes from the result of Takei for the exact WKB analysis of the Weber equation

$$\left(\frac{d^2}{dx^2} - \eta^2\left(E - \frac{1}{4}x^2\right)\right)\psi = 0, \quad (1.2)$$

which is a linear ordinary differential equation having two simple turning points at $x = \pm 2\sqrt{E}$ if $E \neq 0$. A degeneration of Stokes geometry of (1.2) occurs when $\arg E = 0$. In [T2] Takei investigated the parametric Stokes phenomenon (with respect to the parameter E) of (1.2) and obtained a connection formula for WKB solutions of (1.2) ([T2, Theorem 2.1]). According to [T2], the parametric Stokes phenomenon is caused by some singularities of the Borel transform of WKB solutions, and furthermore, these singularities originate from the singularities of the Borel transform of the “Voros coefficient” of (1.2). The Voros coefficient is a formal power series of η^{-1} defined by some integral (cf. [T2]). The explicit representation of the Voros coefficient of (1.2) (see (3.23)) was first conjectured by Mikio Sato, and a proof of it based on the use of the creation operator of (1.2) was given by Takei in [T2]. The connection formula for the parametric Stokes phenomenon was obtained as a corollary of it. In this paper we extend the analysis presented in [T2] to (P_{II}) .

Having the results of [T2] for (1.2) in mind, we will introduce the “Voros coefficient of (P_{II}) ” (or the “ P -Voros coefficient”, for short) in Section 3.1, and obtain the explicit representation of it in Section 3.2. The following is one of the main results of this paper:

Theorem 1.1 (Theorem 3.1). *The P -Voros coefficient $W(c, \eta)$ is represented explicitly as follows:*

$$W(c, \eta) = - \sum_{n=1}^{\infty} \frac{2^{1-2n} - 1}{2n(2n-1)} B_{2n}(c\eta)^{1-2n}, \quad (1.3)$$

where B_{2n} is the $2n$ -th Bernoulli number defined by

$$\frac{w}{e^w - 1} = 1 - \frac{w}{2} + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} w^{2n}. \quad (1.4)$$

Using this expression, we can derive the following connection formula for the parametric Stokes phenomenon under the assumption of Borel summability of 1-parameter solutions:

Connection formula for 1-parameter solutions of (P_{II}) . *Let ε be a sufficiently small positive number.*

(i) *If the true solutions represented by $\lambda_{\infty}(t, c, \eta; \alpha)$ for $\arg c = \frac{\pi}{2} - \varepsilon$ and those by $\lambda_{\infty}(t, c, \eta; \tilde{\alpha})$ for $\arg c = \frac{\pi}{2} + \varepsilon$ coincide, then the following holds:*

$$\tilde{\alpha} = \alpha. \quad (1.5)$$

(ii) *If the true solutions represented by $\lambda_{\tau}(t, c, \eta; \alpha)$ for $\arg c = \frac{\pi}{2} - \varepsilon$ and those by $\lambda_{\tau}(t, c, \eta; \tilde{\alpha})$ for $\arg c = \frac{\pi}{2} + \varepsilon$ coincide, then the following holds:*

$$\tilde{\alpha} = (1 + e^{2\pi i c \eta}) \alpha. \quad (1.6)$$

On the other hand, we can also derive the connection formula for the parametric Stokes phenomenon by using the isomonodromic deformation of (SL_{II}) , which is a second order linear ordinary differential equation relevant to (P_{II}) :

$$(SL_{\text{II}}) : \left(\frac{\partial^2}{\partial x^2} - \eta^2 Q_{\text{II}}\right) \psi = 0,$$

where

$$Q_{\text{II}} = x^4 + tx^2 + 2cx + 2K_{\text{II}} - \eta^{-1} \frac{\nu}{x - \lambda} + \eta^{-2} \frac{3}{4(x - \lambda)^2},$$

$$K_{\text{II}} = \frac{1}{2}[\nu^2 - (\lambda^4 + t\lambda^2 + 2c\lambda)].$$

We compute the Stokes multipliers of (SL_{II}) around $x = \infty$ by using the “Iso-Monodromic” WKB solutions $\psi_{\pm, \text{IM}}$, which satisfy both (SL_{II}) and its deformation equation (D_{II}) . (The construction of $\psi_{\pm, \text{IM}}$ is explained in Section 4.2.) Let \mathfrak{s}_j (resp. \mathfrak{s}_j') be the Stokes multipliers computed by using $\psi_{\pm, \text{IM}}$ when $\arg c = \frac{\pi}{2} - \varepsilon$ (resp. $\arg c = \frac{\pi}{2} + \varepsilon$) with a sufficiently small positive number ε ($1 \leq j \leq 6$). The results of the computations (those are presented in Section 6.3) are given by the following lists:

Stokes multipliers of (SL_{II}) around $x = \infty$.

(i) If the 1-parameter solution substituted into the coefficients of (SL_{II}) and (D_{II}) is normalized at ∞ , the Stokes multipliers of (SL_{II}) are given by the following:

$$\begin{cases} \mathfrak{s}_1 = i (1 + e^{2\pi i c \eta}) e^{U-2V} \\ \mathfrak{s}_2 = i e^{-2\pi i c \eta} e^{2V-U} \\ \mathfrak{s}_3 = i (1 + e^{2\pi i c \eta}) e^{-2\pi i c \eta} e^{U-2V} \\ \mathfrak{s}_4 = -2\sqrt{\pi} \alpha \\ \mathfrak{s}_5 = 0 \\ \mathfrak{s}_6 = 2\sqrt{\pi} \alpha + i e^{2V-U}. \end{cases} \quad \begin{cases} \mathfrak{s}'_1 = i e^{U-2V} \\ \mathfrak{s}'_2 = i (1 + e^{2\pi i c \eta}) e^{-2\pi i c \eta} e^{2V-U} \\ \mathfrak{s}'_3 = i e^{-2\pi i c \eta} e^{U-2V} \\ \mathfrak{s}'_4 = -2\sqrt{\pi} \alpha \\ \mathfrak{s}'_5 = 0 \\ \mathfrak{s}'_6 = 2\sqrt{\pi} \alpha + i (1 + e^{2\pi i c \eta}) e^{2V-U}. \end{cases} \quad (1.7)$$

(ii) If the 1-parameter solution substituted into the coefficients of (SL_{II}) and (D_{II}) is normalized at τ , the Stokes multipliers of (SL_{II}) are given by the following:

$$\begin{cases} \mathfrak{s}_1 = i (1 + e^{2\pi i c \eta}) e^{U-2V} \\ \mathfrak{s}_2 = i e^{-2\pi i c \eta} e^{2V-U} \\ \mathfrak{s}_3 = i (1 + e^{2\pi i c \eta}) e^{-2\pi i c \eta} e^{U-2V} \\ \mathfrak{s}_4 = -2\sqrt{\pi} \alpha e^W \\ \mathfrak{s}_5 = 0 \\ \mathfrak{s}_6 = 2\sqrt{\pi} \alpha e^W + i e^{2V-U}. \end{cases} \quad \begin{cases} \mathfrak{s}'_1 = i e^{U-2V} \\ \mathfrak{s}'_2 = i (1 + e^{2\pi i c \eta}) e^{-2\pi i c \eta} e^{2V-U} \\ \mathfrak{s}'_3 = i e^{-2\pi i c \eta} e^{U-2V} \\ \mathfrak{s}'_4 = -2\sqrt{\pi} \alpha e^W \\ \mathfrak{s}'_5 = 0 \\ \mathfrak{s}'_6 = 2\sqrt{\pi} \alpha e^W + i (1 + e^{2\pi i c \eta}) e^{2V-U}. \end{cases} \quad (1.8)$$

Here α is the free parameter contained in the 1-parameter solution substituted into the coefficients of (SL_{II}) and (D_{II}) , $U = U(t, c, \eta; \alpha)$ is given by the following integral

$$U = \eta \int_{\infty}^t (\lambda(t, c, \eta; \alpha) - \lambda_0^{(0)}(t, c)) dt, \quad (1.9)$$

$V = V(t, c, \eta; \alpha)$ is the Voros coefficient of (SL_{II}) and $W = W(c, \eta)$ is the P -Voros coefficient.

Precisely speaking, the Stokes multipliers are the Borel sums of the formal series in the above lists. They should be independent of t because $\psi_{\pm, \text{IM}}$ satisfies the deformation equation (D_{II}) . This fact can be confirmed by the following theorem, which will be verified through the analysis of the Voros coefficient V of (SL_{II}) in Section 5.

Theorem 1.2 (Theorem 5.1). *The Voros coefficient V of (SL_{II}) and U given by (1.9) are related as follows:*

$$2V(t, c, \eta) - U(t, c, \eta) = - \sum_{n=1}^{\infty} \frac{2^{1-2n} - 1}{2n(2n-1)} B_{2n}(c\eta)^{1-2n}, \quad (1.10)$$

where B_{2n} is the $2n$ -th Bernoulli number defined by (1.4).

If two 1-parameter solutions of (P_{II}) are given and true solutions of (P_{II}) represented by these 1-parameter solutions coincide, then the corresponding Stokes multipliers of (SL_{II}) should coincide. Thus, making use of the expression (1.10), we can derive the connection formulas which describe the parametric Stokes phenomenon occurring to 1-parameter solutions substituted into the coefficients of (SL_{II}) and (D_{II}) . The details will be explained in Section 6.4. Furthermore, the connection formulas obtained in this way coincide with (1.5) and (1.6), that is, the connection formulas (1.5) and (1.6) describing the parametric Stokes phenomenon for 1-parameter solutions can be confirmed also by the isomonodromic deformation method. This is our main claim.

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2 1-parameter solutions and the P -Stokes geometry of (P_{II})

The general theory of the exact WKB analysis of Painlevé equations (P_J) ($J = \text{I}, \dots, \text{VI}$) was developed in a series of papers [KT2], [AKT1], [KT3] and 1-parameter solutions of (P_J) were discussed in [KT1]. (See also [KT4, §4].) In this section, we review the core part of the exact WKB analysis of (P_J) in the case of (P_{II}) .

$$(P_{\text{II}}) : \frac{d^2 \lambda}{dt^2} = \eta^2 (2\lambda^3 + t\lambda + c).$$

2.1 0-parameter solutions and 1-parameter solutions of (H_{II})

(P_{II}) is equivalent to the following Hamiltonian system (H_{II}) :

$$(H_{\text{II}}) : \begin{cases} \frac{d\lambda}{dt} = \eta \nu, \\ \frac{d\nu}{dt} = \eta (2\lambda^3 + t\lambda + c). \end{cases}$$

(H_{II}) has a formal power series solution $(\lambda^{(0)}, \nu^{(0)})$ of η^{-1} called a 0-parameter solution of (H_{II}) .

$$\begin{cases} \lambda^{(0)}(t, c, \eta) = \lambda_0^{(0)}(t, c) + \eta^{-1} \lambda_1^{(0)}(t, c) + \eta^{-2} \lambda_2^{(0)}(t, c) + \dots, \\ \nu^{(0)}(t, c, \eta) = \nu_0^{(0)}(t, c) + \eta^{-1} \nu_1^{(0)}(t, c) + \eta^{-2} \nu_2^{(0)}(t, c) + \dots. \end{cases}$$

Here $\lambda_0^{(0)}$ satisfies

$$2\lambda_0^{(0)3} + t\lambda_0^{(0)} + c = 0 \tag{2.1}$$

and

$$\nu_0^{(0)} = 0. \tag{2.2}$$

In what follows we abbreviate $\lambda_0^{(0)}$ and $\nu_0^{(0)}$ to λ_0 and ν_0 , respectively. Once the branch of λ_0 is fixed, the coefficients $(\lambda_k^{(0)}, \nu_k^{(0)})$ of η^{-k} in $(\lambda^{(0)}, \nu^{(0)})$ are determined by the following recursive relations:

$$(6\lambda_0^2 + t)\lambda_k^{(0)} + 2 \sum_{\substack{k_1 + k_2 + k_3 = k \\ 0 \leq k_j < k}} \lambda_{k_1}^{(0)} \lambda_{k_2}^{(0)} \lambda_{k_3}^{(0)} = \frac{d^2 \lambda_{k-2}^{(0)}}{dt^2} \quad (k \geq 1), \tag{2.3}$$

$$\nu_k^{(0)} = \frac{d\lambda_{k-1}^{(0)}}{dt} \quad (k \geq 1). \quad (2.4)$$

We can also construct a 1-parameter family of formal solutions, called 1-parameter solution, of (H_{II}) of the following form:

$$\begin{cases} \lambda(t, c, \eta; \alpha) = \lambda^{(0)}(t, c, \eta) + \alpha \eta^{-\frac{1}{2}} \lambda^{(1)}(t, c, \eta) e^{\eta \phi_{\text{II}}} + (\alpha \eta^{-\frac{1}{2}})^2 \lambda^{(2)}(t, c, \eta) e^{2\eta \phi_{\text{II}}} + \dots, \\ \nu(t, c, \eta; \alpha) = \nu^{(0)}(t, c, \eta) + \alpha \eta^{-\frac{1}{2}} \nu^{(1)}(t, c, \eta) e^{\eta \phi_{\text{II}}} + (\alpha \eta^{-\frac{1}{2}})^2 \nu^{(2)}(t, c, \eta) e^{2\eta \phi_{\text{II}}} + \dots, \end{cases}$$

where α is a free parameter, $\lambda^{(k)}$ and $\nu^{(k)}$ are formal power series of η^{-1} ,

$$\lambda^{(k)}(t, c, \eta) = \lambda_0^{(k)}(t, c) + \eta^{-1} \lambda_1^{(k)}(t, c) + \eta^{-2} \lambda_2^{(k)}(t, c) + \dots,$$

$$\nu^{(k)}(t, c, \eta) = \nu_0^{(k)}(t, c) + \eta^{-1} \nu_1^{(k)}(t, c) + \eta^{-2} \nu_2^{(k)}(t, c) + \dots,$$

and

$$\phi_{\text{II}}(t, c) = \int^t \sqrt{\Delta(t, c)} dt,$$

$$\Delta(t, c) = 6\lambda_0(t, c)^2 + t.$$

In what follows 1-parameter solutions are considered in a domain where the real part of ϕ_{II} is negative, i.e., $e^{\eta \phi_{\text{II}}}$ is exponentially small when $\eta \rightarrow \infty$.

Here we briefly recall the construction of a 1-parameter solution. (It is similar to the construction of the so-called transseries solution; cf. [C, pp.90-91].) First, $\lambda^{(0)}$ is a 0-parameter solution constructed above. Second, if $\alpha \eta^{-\frac{1}{2}} \lambda^{(1)}(t, c, \eta) e^{\eta \phi_{\text{II}}}$ is denoted by $\tilde{\lambda}^{(1)}$, then $\tilde{\lambda}^{(1)}$ is a solution of the following second order linear differential equation:

$$\frac{d^2 \tilde{\lambda}^{(1)}}{dt^2} = \eta^2 (6\lambda^{(0)}(t, c, \eta)^2 + t) \tilde{\lambda}^{(1)}, \quad (2.5)$$

that is, the Fréchet derivative of (P_{II}) at $\lambda = \lambda^{(0)}$. Thus $\tilde{\lambda}^{(1)}$ can be taken as a WKB solution (see [KT4, §2.1] for example) of (2.5) of the form

$$\tilde{\lambda}^{(1)} = \alpha \frac{1}{\sqrt{R_{\text{odd}}(t, c, \eta)}} \exp\left(\int^t R_{\text{odd}}(t, c, \eta) dt\right) \quad (2.6)$$

$$= \alpha \eta^{-\frac{1}{2}} (\lambda_0^{(1)}(t, c) + \eta^{-1} \lambda_1^{(1)}(t, c) + \eta^{-2} \lambda_2^{(1)}(t, c) + \dots) e^{\eta \phi_{\text{II}}}, \quad (2.7)$$

where R_{odd} is the odd part (in the sense of Remark 2.1 below) of a formal power series solution $R = \eta R_{-1} + R_0 + \eta^{-1} R_1 + \dots$ of the Riccati equation

$$R^2 + \frac{dR}{dt} = \eta^2 (6\lambda^{(0)}(t, c, \eta)^2 + t) \quad (2.8)$$

associated with (2.5).

Remark 2.1. The odd part of R is defined as follows. The coefficients R_k of η^{-k} in R are determined by the following recursive relations:

$$\begin{aligned} R_{-1}^2 &= \Delta = 6\lambda_0^2 + t, \\ 2R_{-1}R_{k+1} + \sum_{\substack{k_1 + k_2 = k \\ 0 \leq k_j}} R_{k_1}R_{k_2} + \frac{dR_k}{dt} &= 6 \sum_{\substack{l_1 + l_2 = k+2 \\ 0 \leq l_j}} \lambda_{l_1}^{(0)} \lambda_{l_2}^{(0)} \quad (k \geq -1). \end{aligned} \quad (2.9)$$

Once the branch of

$$R_{-1} = \sqrt{\Delta} \quad (2.10)$$

is fixed, R_k ($k \geq 0$) is determined uniquely by (2.9). Hence we obtain a formal solution $R = R(t, c, \eta)$ of (2.8). Similarly we obtain a formal solution $R^\dagger = R^\dagger(t, c, \eta)$ of (2.8), starting with $R_{-1}^\dagger = -\sqrt{\Delta}$. Then, we define the odd and even part of R by

$$R_{\text{odd}}(t, c, \eta) = \frac{1}{2}(R(t, c, \eta) - R^\dagger(t, c, \eta)), \quad (2.11)$$

$$R_{\text{even}}(t, c, \eta) = \frac{1}{2}(R(t, c, \eta) + R^\dagger(t, c, \eta)). \quad (2.12)$$

We also note that, as in [KT4, §2.1],

$$R_{\text{even}} = -\frac{1}{2} \frac{1}{R_{\text{odd}}} \frac{dR_{\text{odd}}}{dt} = -\frac{1}{2} \frac{d}{dt} \log R_{\text{odd}}. \quad (2.13)$$

An important fact is that, once we fix a normalization (i.e., the lower endpoint) of the integral of R_{odd} in (2.6), the coefficients $\lambda_\ell^{(k)}$ of $\lambda^{(k)}$ ($k \geq 2$, $\ell \geq 0$) are determined uniquely by the following recursive relations.

$$\begin{aligned} (k^2 - 1)\Delta(t, c) \lambda_\ell^{(k)} &= 6 \sum_{\substack{\ell_1 + \ell_2 + \ell_3 = \ell \\ \ell_3 < \ell}} \lambda_{\ell_1}^{(0)} \lambda_{\ell_2}^{(0)} \lambda_{\ell_3}^{(k)} \\ &\quad + 2 \sum_{\substack{k_1 + k_2 + k_3 = k \\ \ell_1 + \ell_2 + \ell_3 = \ell \\ k_j < k}} \lambda_{\ell_1}^{(k_1)} \lambda_{\ell_2}^{(k_2)} \lambda_{\ell_3}^{(k_3)} \\ &\quad - 2k \frac{d\phi_{\text{II}}}{dt} \frac{d\lambda_{\ell-1}^{(k)}}{dt} - k \frac{d^2\phi_{\text{II}}}{dt^2} \lambda_{\ell-1}^{(k)} - \frac{d^2\lambda_{\ell-2}^{(k)}}{dt^2}. \end{aligned} \quad (2.14)$$

Thus we can construct a 1-parameter solution $\lambda(t, c, \eta; \alpha)$ including a free parameter α . (Normalization of the integral in (2.6) will be discussed in Section 3.) Since $\nu = \eta^{-1} \frac{d\lambda}{dt}$ follows from (H_{II}) , the formal power series $\nu^{(k)}$ ($k \geq 0$) are determined by

$$\nu^{(k)} = k \frac{d\phi_{\text{II}}}{dt} \lambda^{(k)} + \eta^{-1} \frac{d\lambda^{(k)}}{dt}. \quad (2.15)$$

Especially, since $\tilde{\lambda}^{(1)} = \alpha \eta^{-\frac{1}{2}} \lambda^{(1)} e^{\eta\phi_{\text{II}}}$ can be written also as

$$\tilde{\lambda}^{(1)} = \alpha \eta^{-\frac{1}{2}} C(\eta) \exp\left(\int^t R(t, c, \eta) dt\right)$$

with a formal power series $C(\eta)$ of η^{-1} whose coefficients are independent of t , we have

$$\nu^{(1)} = \eta^{-1} R \lambda^{(1)}. \quad (2.16)$$

2.2 P -Stokes geometry of (P_{II})

Next, we recall the definition of turning points (“ P -turning points”) and Stokes curves (“ P -Stokes curves”) of (P_{II}) .

Definition 2.1 ([KT4, §4, Definition 4.5]). (i) A point t is called a P -turning point of (P_{II}) if t satisfies $\Delta = 6\lambda_0^2 + t = 0$.

(ii) For a P -turning point $t = \tau$, a real one-dimensional curve defined by

$$\text{Im} \int_{\tau}^t \sqrt{\Delta(t, c)} dt = 0$$

is said to be a P -Stokes curve of (P_{II}) .

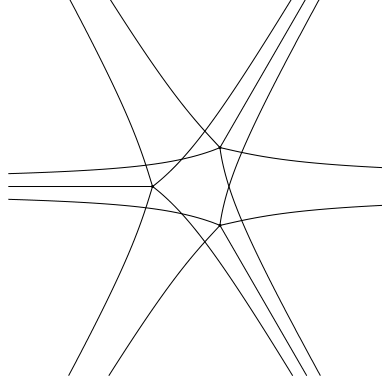


Figure 2.1: P -Stokes curve when $\arg c = 0$.

The P -turning points and the P -Stokes curves are the turning points and the Stokes curves of the linear equation (2.5). Since P -turning points are also zeros of discriminant of the algebraic equation (2.1) for λ_0 , there are three P -turning points at $t = \tau_j := -6(c/4)^{2/3} \omega^j$ ($\omega = e^{2\pi i/3}, j = 1, 2, 3$) in the case $c \neq 0$. Figure 2.1 describes P -Stokes curves when $\arg c = 0$. As we saw in Section 1, some degeneration of P -Stokes geometry occurs when $\arg c = \frac{\pi}{2}$ (Figure 1.1 ~ 1.3). This degeneracy can be analytically confirmed by the relation

$$\int_{\tau_1}^{\tau_2} \sqrt{\Delta} dt = \pm 2\pi ic,$$

which we will show in Proposition 4.1 in Section 4. (The choice of the sign on the right-hand side of the above relation depends on the determination of branch of $\sqrt{\Delta}$.)

Remark 2.2. Since P -turning points and P -Stokes curves are defined in terms of $\Delta = 6\lambda_0^2 + t$, it is natural to lift them onto the Riemann surface of λ_0 . Figure 2.2 describes the lift of P -Stokes curves onto the Riemann surface of λ_0 when $\arg c = \frac{\pi}{2}$. Wiggly lines, solid

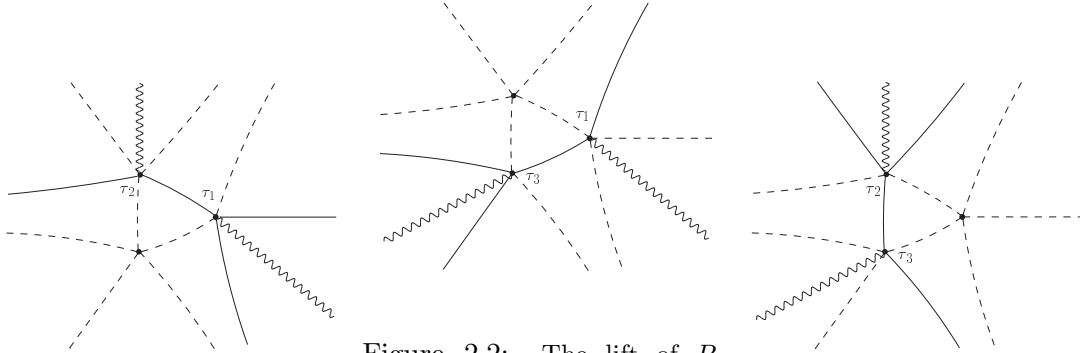


Figure 2.2: The lift of P -Stokes curves onto the Riemann surface of λ_0

lines and dotted lines in Figure 2.2 represent cuts to define the Riemann surface of λ_0 , P -Stokes curves on the sheet under consideration and P -Stokes curves on the other sheets, respectively. In this paper we only consider the situation where $\arg c$ is sufficiently close to $\frac{\pi}{2}$ and t moves in the shaded domain in Figure 2.3 below. In Figure 2.3 thick wiggly lines designate cuts for the determination of the branch of $\sqrt{\Delta}$ and the symbols \oplus and \ominus represent the “sign of P -Stokes curves”. Here the sign of a P -Stokes curve is defined by the sign of

$$\operatorname{Re} \int_{\tau}^t \sqrt{\Delta} dt,$$

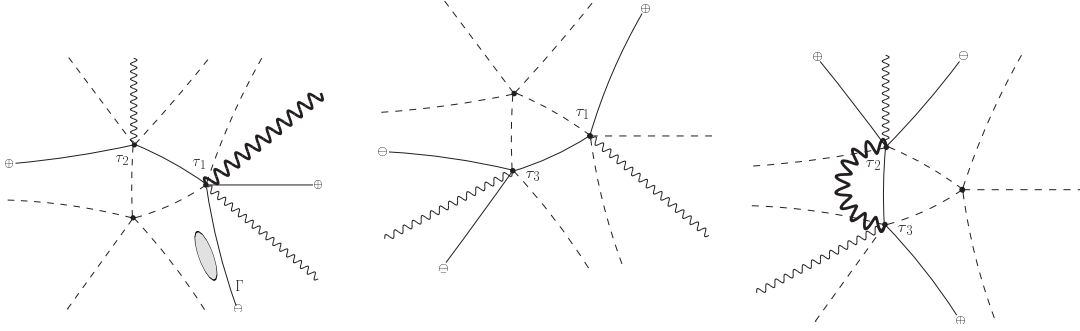


Figure 2.3: Domain of t .

where t is a point on the P -Stokes curve in question and τ is a P -turning point which the P -Stokes curve emanates from.

As we mentioned in Introduction, the degeneration of P -Stokes geometry observed when $\arg c = \frac{\pi}{2}$ suggests that the “parametric Stokes phenomenon” occurs when c varies near $\arg c = \frac{\pi}{2}$. To formulate the connection formula for this Stokes phenomenon, we define and analyze “the Voros coefficient of (P_{II}) ” in the next section. The Voros coefficient plays an important role in the analysis of the parametric Stokes phenomenon.

The following lemma will be used in the next section.

Lemma 2.1. *We have the following asymptotic behaviors when $t \rightarrow \infty$ along the P -Stokes curve Γ in Figure 2.3.*

$$\lambda_0(t, c) = -\frac{i}{\sqrt{2}}t^{\frac{1}{2}} + \frac{1}{2}ct^{-1} - \frac{3\sqrt{2}i}{8}c^2t^{-\frac{5}{2}} + O(t^{-4}). \quad (2.17)$$

$$\lambda_2^{(0)}(t, c) = -\frac{\sqrt{2}i}{16}t^{-\frac{5}{2}} + O(t^{-4}). \quad (2.18)$$

$$\lambda_{2k}^{(0)}(t, c) = O(t^{-\frac{11}{2}}) \quad (k \geq 2). \quad (2.19)$$

$$\nu_1^{(0)}(t, c) = -\frac{i}{2\sqrt{2}}t^{-\frac{1}{2}} - \frac{1}{2}ct^{-2} + \frac{15\sqrt{2}i}{16}c^2t^{-\frac{7}{2}} + O(t^{-5}). \quad (2.20)$$

$$\nu_3^{(0)}(t, c) = \frac{5\sqrt{2}i}{32}t^{-\frac{7}{2}} + O(t^{-5}). \quad (2.21)$$

$$\nu_{2k+1}^{(0)}(t, c) = O(t^{-\frac{13}{2}}) \quad (k \geq 2). \quad (2.22)$$

$$R_{-1}(t, c) = -\sqrt{2}it^{\frac{1}{2}} + \frac{3}{2}ct^{-1} - \frac{21\sqrt{2}i}{16}c^2t^{-\frac{5}{2}} + O(t^{-4}). \quad (2.23)$$

$$R_0(t, c) = -\frac{1}{4}t^{-1} + \frac{9\sqrt{2}i}{16}ct^{-\frac{5}{2}} + O(t^{-4}). \quad (2.24)$$

$$R_1(t, c) = -\frac{17\sqrt{2}i}{64}t^{-\frac{5}{2}} + O(t^{-4}). \quad (2.25)$$

$$R_k(t, c) = O(t^{-4}) \quad (k \geq 2). \quad (2.26)$$

Here the branch of $t^{\frac{1}{2}}$ is chosen so that $\operatorname{Re} t^{\frac{1}{2}} > 0$ holds on Γ .

Proof. It follows from (2.1) that λ_0 has the following three possible asymptotic behaviors

when $t \rightarrow \infty$.

$$\lambda_0 \sim \begin{cases} +\frac{i}{\sqrt{2}}t^{\frac{1}{2}}, \\ -\frac{i}{\sqrt{2}}t^{\frac{1}{2}}, \\ -ct^{-1}. \end{cases}$$

Especially, the behavior of λ_0 when $t \rightarrow \infty$ along the P -Stokes curve Γ is given by

$$\lambda_0 \sim -\frac{i}{\sqrt{2}}t^{\frac{1}{2}}.$$

Thus we have (2.17). By (2.17) we find that $R_{-1} = \sqrt{\Delta}$ has two possible asymptotic behaviors below.

$$R_{-1} \sim \begin{cases} +\sqrt{2}it^{\frac{1}{2}}, \\ -\sqrt{2}it^{\frac{1}{2}}. \end{cases}$$

Because the sign of the P -Stokes curve Γ is \ominus , we have

$$R_{-1} \sim -\sqrt{2}it^{\frac{1}{2}}.$$

Thus we obtain (2.23). The other asymptotic behaviors are obtained from (2.17), (2.23) and the recursive relations (cf. (2.3), (2.4) and (2.9)). \square

3 The Voros coefficient and the parametric Stokes phenomena of (P_{II})

To formulate the connection formula for the 1-parameter solutions of (P_{II}) , first we introduce two normalizations of 1-parameter solutions. The Voros coefficient of (P_{II}) is defined as the difference of these two normalizations.

3.1 The Voros coefficient of (P_{II})

We introduce two normalizations of the integral

$$\int_{\tau_1}^t R_{\text{odd}} dt$$

in (2.6). Because the coefficients R_{2k-1} of $\eta^{-(2k-1)}$ in R_{odd} have a singularity of the form $(t - \tau_1)^{-\frac{l}{4}}$ (where l is an odd integer) at a P -turning point $t = \tau_1$, we can define the integral of R_{odd} from $t = \tau_1$ as a contour integral:

$$\tilde{\lambda}_{\tau_1}^{(1)}(t, c, \eta; \alpha) = \alpha \frac{1}{\sqrt{R_{\text{odd}}}} \exp\left(\int_{\tau_1}^t R_{\text{odd}} dt\right), \quad (3.1)$$

$$\lambda_{\tau_1}^{(1)}(t, c, \eta) = \Delta^{-\frac{1}{4}} + \eta^{-1} \Delta^{-\frac{1}{4}} \left(\int_{\tau_1}^t R_1 dt\right) + \cdots. \quad (3.2)$$

Here the integral $\int_{\tau_1}^t R_{\text{odd}} dt$ in (3.1) is defined by $\frac{1}{2} \int_{\Gamma_t} R_{\text{odd}} dt$, where Γ_t is a path on the Riemann surface of $\sqrt{\Delta}$ shown in Figure 3.1 and \check{t} represents a point on the Riemann surface of $\sqrt{\Delta}$ satisfying that $\lambda_0(\check{t}, c) = \lambda_0(t, c)$ and $\sqrt{\Delta(\check{t}, c)} = -\sqrt{\Delta(t, c)}$. (The dotted part of Γ_t represents a path on the other sheet of the Riemann surface of $\sqrt{\Delta}$.) On the

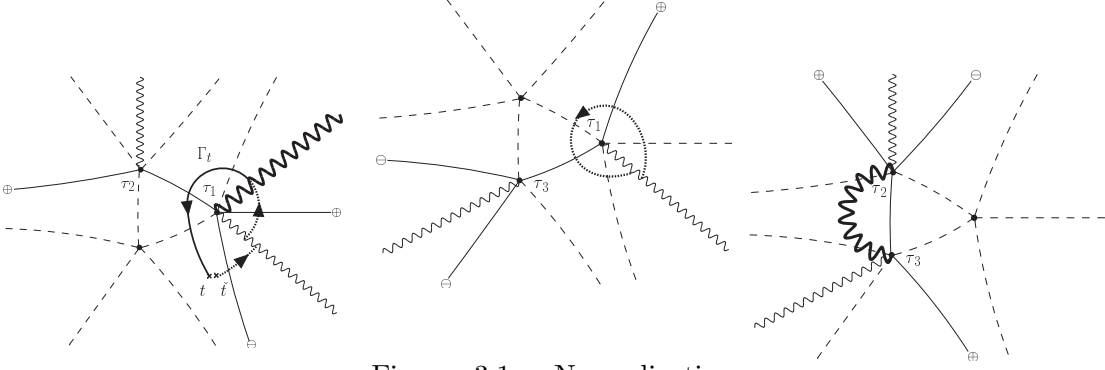


Figure 3.1: Normalization path Γ_t of $\tilde{\lambda}_{\tau_1}^{(1)}$.

other hand, since R_{2k-1} ($k \geq 1$) are integrable at $t = \infty$ by Lemma 2.1, the following integral is well-defined:

$$\tilde{\lambda}_{\infty}^{(1)}(t, c, \eta; \alpha) = \alpha \frac{1}{\sqrt{R_{\text{odd}}}} \exp \left(\eta \int_{\tau_1}^t R_{-1} dt + \int_{\infty}^t (R_{\text{odd}} - \eta R_{-1}) dt \right), \quad (3.3)$$

$$\lambda_{\infty}^{(1)}(t, c, \eta) = \Delta^{-\frac{1}{4}} + \eta^{-1} \Delta^{-\frac{1}{4}} \left(\int_{\infty}^t R_1 dt \right) + \dots, \quad (3.4)$$

where the path of integral from infinity is taken as in Figure 3.2. As we noted in Section 2,

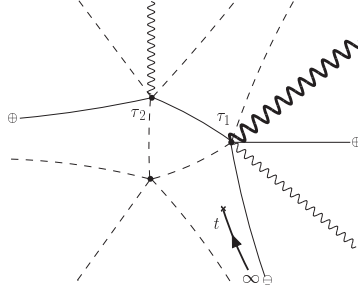


Figure 3.2: Normalization path of $\tilde{\lambda}_{\infty}^{(1)}$.

once the normalization of $\tilde{\lambda}^{(1)}$ is fixed, a 1-parameter solution is uniquely determined. We define the 1-parameter solution $\lambda_{\tau_1}(t, c, \eta; \alpha)$ (resp. $\lambda_{\infty}(t, c, \eta; \alpha)$) by using $\tilde{\lambda}_{\tau_1}^{(1)}$ (resp. $\tilde{\lambda}_{\infty}^{(1)}$) for the normalization of $\tilde{\lambda}^{(1)}$. From now on we consider 1-parameter solutions normalized at either $t = \tau_1$ or $t = \infty$. Therefore, ϕ_{II} is always normalized as

$$\phi_{\text{II}} = \int_{\tau_1}^t \sqrt{\Delta} dt.$$

Next, we define the Voros coefficient of (P_{II}) . There is a relation between the above two normalizations of $\tilde{\lambda}^{(1)}$:

$$\tilde{\lambda}_{\tau_1}^{(1)} = e^W \tilde{\lambda}_{\infty}^{(1)}, \quad (3.5)$$

where

$$\begin{aligned} W = W(c, \eta) &= \int_{\tau_1}^{\infty} (R_{\text{odd}}(t, c, \eta) - \eta R_{-1}(t, c)) dt \\ &= \frac{1}{2} \int_{\Gamma_{\infty}} (R_{\text{odd}}(t, c, \eta) - \eta R_{-1}(t, c)) dt. \end{aligned} \quad (3.6)$$

Here Γ_{∞} is a path on the Riemann surface of $\sqrt{\Delta}$ shown in Figure 3.3.

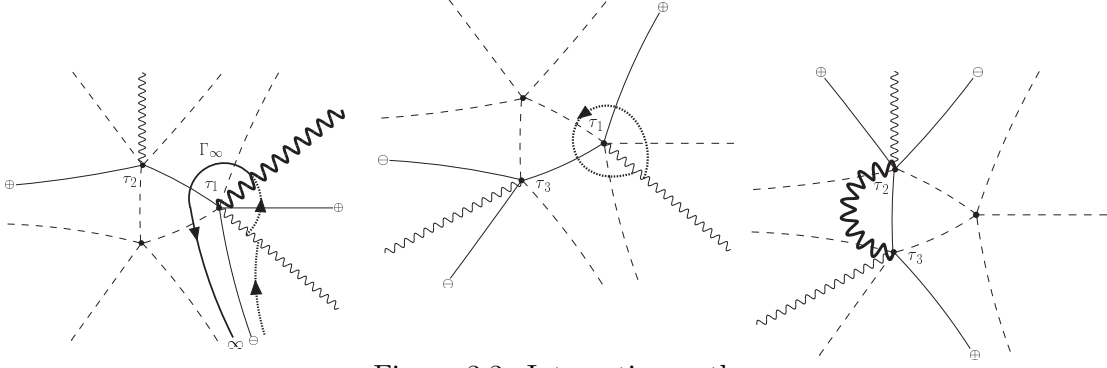


Figure 3.3: Integration path Γ_∞ .

Definition 3.1. $W(c, \eta)$ defined by (3.6) is called *the Voros coefficient of (P_{II})* , or the *P-Voros coefficient* for short.

Remark 3.1. By the uniqueness of $\lambda^{(k)}$ ($k \geq 2$), the relation (3.5) induces the relation between the two normalizations of 1-parameter solutions as follows:

$$\lambda_{\tau_1}(t, c, \eta; \alpha) = \lambda_\infty(t, c, \eta; \alpha e^W). \quad (3.7)$$

3.2 Determination of the Voros coefficient of (P_{II})

We have an explicit description of the *P-Voros coefficient*.

Theorem 3.1. *The P-Voros coefficient (3.6) is represented as follows:*

$$W(c, \eta) = - \sum_{n=1}^{\infty} \frac{2^{1-2n} - 1}{2n(2n-1)} B_{2n}(c\eta)^{1-2n}, \quad (3.8)$$

where B_{2n} is the $2n$ -th Bernoulli number defined by (1.4).

In what follows we prove this theorem by using the idea of [T2].

Lemma 3.1 ([T2, Lemma 1.2]). (i) *The formal power series*

$$F(c, \eta) = - \sum_{n=1}^{\infty} \frac{2^{1-2n} - 1}{2n(2n-1)} B_{2n}(c\eta)^{1-2n} \quad (3.9)$$

satisfies the following difference equation formally:

$$F(c, \eta) - F(c - \eta^{-1}, \eta) = -1 + c\eta \log\left(1 + \frac{1}{c\eta - 1}\right) - \log\left(1 + \frac{1}{2(c\eta - 1)}\right). \quad (3.10)$$

(ii) *Conversely, if $F(c, \eta) = \sum_{n=1}^{\infty} F_n(c\eta)^{-n}$ where $F_n \in \mathbb{C}$ is independent of c is a formal solution of (3.10), then it coincides with (3.9).*

By (A.14) in Appendix,

$$W(r^{-1}c, r\eta) = W(c, \eta) \quad (3.11)$$

holds for all positive real numbers r . Thus $W(c, \eta)$ is written in the form $\sum_{n=1}^{\infty} W_n(c\eta)^{-n}$ and W_n ($n \geq 1$) is independent of c . Hence it is sufficient to show that $W(c, \eta)$ satisfies the difference equation (3.10).

Lemma 3.2.

$$W(c, \eta) = \frac{1}{2} \int_{\Gamma_\infty} (R(t, c, \eta) - \eta R_{-1}(t, c) - R_0(t, c)) dt.$$

Proof. We have to prove that the following relation holds:

$$\int_{\Gamma_\infty} R_{\text{even}}(t, c, \eta) dt = \int_{\Gamma_\infty} R_0(t, c) dt.$$

By the relation (2.13), we obtain

$$\begin{aligned} R_{\text{even}} &= -\frac{1}{2} \frac{d}{dt} \log R_{\text{odd}} \\ &= -\frac{1}{2R_{-1}} \frac{dR_{-1}}{dt} - \frac{1}{2} \frac{d}{dt} \log \left(1 + \frac{R_{\text{odd}} - \eta R_{-1}}{\eta R_{-1}} \right). \end{aligned}$$

Since the first term of the right-hand side coincides with R_0 and the integral of the second term along Γ_∞ vanishes, we obtain the desired relation. \square

By Lemma 3.2, if we define

$$\begin{aligned} I(t, c, \eta) &= \int_{\Gamma_t} R(t, c, \eta) dt - \int_{\Gamma_t} R(t, c - \eta^{-1}, \eta) dt, \\ I_j(t, c, \eta) &= \int_{\Gamma_t} R_j(t, c) dt - \int_{\Gamma_t} R_j(t, c - \eta^{-1}) dt \quad (j \geq -1), \end{aligned}$$

then we obtain

$$W(c, \eta) - W(c - \eta^{-1}, \eta) = \frac{1}{2} \lim_{t \rightarrow \infty} (I(t, c, \eta) - \eta I_{-1}(t, c, \eta) - I_0(t, c, \eta)), \quad (3.12)$$

where the limit is taken along the P -Stokes curve Γ in Figure 2.3. To calculate the right-hand side of (3.12), we employ the so-called Bäcklund transformation which induces the translation of the parameter $c \mapsto c - \eta^{-1}$.

Lemma 3.3 ([JM, pp.423-424]). *Let (λ, ν) be a solution of (H_{II}) . If we define*

$$\begin{cases} \Lambda(\lambda, \nu) = -\lambda + \frac{c - \frac{1}{2}\eta^{-1}}{\nu - \lambda^2 - \frac{1}{2}t}, \\ \mathcal{N}(\lambda, \nu) = -\nu + \frac{2(c - \frac{1}{2}\eta^{-1})\lambda}{\nu - \lambda^2 - \frac{1}{2}t} - \left(\frac{c - \frac{1}{2}\eta^{-1}}{\nu - \lambda^2 - \frac{1}{2}t} \right)^2, \end{cases} \quad (3.13)$$

then $(\Lambda, \mathcal{N}) = (\Lambda(\lambda, \nu), \mathcal{N}(\lambda, \nu))$ satisfies the following Hamiltonian system:

$$\begin{cases} \frac{d\Lambda}{dt} = \eta \mathcal{N}, \\ \frac{d\mathcal{N}}{dt} = \eta (2\Lambda^3 + t\Lambda + c - \eta^{-1}). \end{cases} \quad (3.14)$$

Thus $\Lambda = \Lambda(\lambda, \nu)$ satisfies

$$\frac{d^2\Lambda}{dt^2} = \eta^2 (2\Lambda^3 + t\Lambda + c - \eta^{-1}). \quad (3.15)$$

This lemma is easily confirmed by straightforward computations. With the aid of Lemma 3.3, we can calculate the difference $R(t, c, \eta) - R(t, c - \eta^{-1}, \eta)$ in the following manner.

Lemma 3.4. *Let $(\lambda^{(0)}, \nu^{(0)}) = (\lambda^{(0)}(t, c, \eta), \nu^{(0)}(t, c, \eta))$ be a 0-parameter solution of (H_{II}) . (i)*

$$\Lambda(\lambda^{(0)}(t, c, \eta), \nu^{(0)}(t, c, \eta)) = \lambda^{(0)}(t, c - \eta^{-1}, \eta), \quad (3.16)$$

$$\mathcal{N}(\lambda^{(0)}(t, c, \eta), \nu^{(0)}(t, c, \eta)) = \nu^{(0)}(t, c - \eta^{-1}, \eta). \quad (3.17)$$

(ii) The formal solution $R(t, c, \eta)$ of (2.8) satisfies the following:

$$R(t, c, \eta) - R(t, c - \eta^{-1}, \eta) = -\frac{d}{dt} \log \left\{ 1 + \left(c - \frac{1}{2} \eta^{-1} \right) \frac{\eta^{-1} R(t, c, \eta) - 2\lambda^{(0)}(t, c, \eta)}{(\nu^{(0)}(t, c, \eta) - \lambda^{(0)}(t, c, \eta)^2 - \frac{1}{2}t)^2} \right\}. \quad (3.18)$$

Proof. (i) By Lemma 3.3, $(\Lambda(\lambda^{(0)}, \nu^{(0)}), \mathcal{N}(\lambda^{(0)}, \nu^{(0)}))$ is a formal power series solution of (3.14), so it is a 0-parameter solution of (3.14). On the other hand, $(\lambda^{(0)}(t, c - \eta^{-1}, \eta), \nu^{(0)}(t, c - \eta^{-1}, \eta))$ is also a 0-parameter solution of (3.14). Because it follows from (2.1) and (2.2) that the leading terms of $\Lambda(\lambda^{(0)}, \nu^{(0)})$ and $\lambda^{(0)}(t, c - \eta^{-1}, \eta)$ both coincide with $\lambda_0(t, c)$, we obtain the relations (3.16) and (3.17) due to the uniqueness of the coefficients of η^{-k} ($k \geq 1$) of 0-parameter solutions of (3.14).

(ii) We apply the Bäcklund transformation (3.13) to a 1-parameter solution $(\lambda(t, c, \eta; \alpha), \nu(t, c, \eta; \alpha))$ of (H_{II}) . Since $\Lambda(\lambda, \nu)$ is expanded as

$$\Lambda(\lambda, \nu) = \Lambda(\lambda^{(0)}, \nu^{(0)}) + \alpha \eta^{-\frac{1}{2}} \left\{ -\lambda^{(1)} - \left(c - \frac{1}{2} \eta^{-1} \right) \frac{\nu^{(1)} - 2\lambda^{(0)}\lambda^{(1)}}{(\nu^{(0)} - \lambda^{(0)^2} - \frac{1}{2}t)^2} \right\} e^{\eta\phi_{II}} + \dots,$$

$(\Lambda(\lambda, \nu), \mathcal{N}(\lambda, \nu))$ is a 1-parameter solution of (3.14). Hence

$$\varphi^{(1)} = \alpha \eta^{-\frac{1}{2}} \left\{ -\lambda^{(1)} - \left(c - \frac{1}{2} \eta^{-1} \right) \frac{\nu^{(1)} - 2\lambda^{(0)}\lambda^{(1)}}{(\nu^{(0)} - \lambda^{(0)^2} - \frac{1}{2}t)^2} \right\} e^{\eta\phi_{II}}$$

is a WKB solution of the linear differential equation obtained as the Fréchet derivative of (3.15) at $\Lambda = \Lambda(\lambda^{(0)}, \nu^{(0)}) = \lambda^{(0)}(t, c - \eta^{-1}, \eta)$;

$$\left(\frac{d^2}{dt^2} - \eta^2 (6\lambda^{(0)}(t, c - \eta^{-1}, \eta)^2 + t) \right) \varphi^{(1)} = 0.$$

This implies that $\varphi^{(1)}$ can be written as $\alpha \eta^{-\frac{1}{2}} C(\eta) \exp\left(\int^t R(t, c - \eta^{-1}, \eta) dt\right)$ for some formal power series $C(\eta)$ of η^{-1} whose coefficients are independent of t . On the other hand, by (2.16), $\varphi^{(1)}$ is also expressed as

$$\alpha \eta^{-\frac{1}{2}} \tilde{C}(\eta) \left\{ 1 + \left(c - \frac{1}{2} \eta^{-1} \right) \frac{\eta^{-1} R(t, c, \eta) - 2\lambda^{(0)}(t, c, \eta)}{(\nu^{(0)}(t, c, \eta) - \lambda^{(0)}(t, c, \eta)^2 - \frac{1}{2}t)^2} \right\} \exp\left(\int^t R(t, c, \eta) dt\right)$$

for some formal series $\tilde{C}(\eta)$ of η^{-1} whose coefficients are independent of t . Therefore, by taking the logarithmic derivative with respect to t of these two different expressions of $\varphi^{(1)}$, we obtain the relation (3.18). \square

By (3.18), $I(t, c, \eta)$ is represented as

$$\begin{aligned} I(t, c, \eta) &= -\log \left\{ 1 + \left(c - \frac{1}{2} \eta^{-1} \right) \frac{\eta^{-1} R(t, c, \eta) - 2\lambda^{(0)}(t, c, \eta)}{(\nu^{(0)}(t, c, \eta) - \lambda^{(0)}(t, c, \eta)^2 - \frac{1}{2}t)^2} \right\} \\ &\quad + \log \left\{ 1 + \left(c - \frac{1}{2} \eta^{-1} \right) \frac{\eta^{-1} R(\check{t}, c, \eta) - 2\lambda^{(0)}(t, c, \eta)}{(\nu^{(0)}(t, c, \eta) - \lambda^{(0)}(t, c, \eta)^2 - \frac{1}{2}t)^2} \right\}. \end{aligned}$$

Using Lemma 2.1 in Section 2 and $R_k(\check{t}, c) = (-1)^k R_k(t, c)$ ($k \geq -1$), we then obtain the following asymptotic behavior of $I(t, c, \eta)$ when taking the limit $t \rightarrow \infty$ along the P -Stokes curve Γ in Figure 2.3:

$$I(t, c, \eta) = -\log \left(-\frac{(1 - 2c\eta)^2}{128} \eta^{-2} t^{-3} \right) + O(t^{-1}). \quad (3.19)$$

Because R_{-1} can be integrated explicitly as

$$\int_{\Gamma_t} R_{-1}(t, c) dt = \frac{4}{3} t \sqrt{\Delta(t, c)} - 2c \log \left\{ \frac{2\lambda_0(t, c) - \sqrt{\Delta(t, c)}}{2\lambda_0(t, c) + \sqrt{\Delta(t, c)}} \right\},$$

we also have the following asymptotic behavior of $I_{-1}(t, c, \eta)$:

$$\eta I_{-1}(t, c, \eta) = 2 + 2c\eta \log \left(\frac{c\eta - 1}{c\eta} \right) + \log \left(-\frac{32}{(c\eta - 1)^2} \eta^2 t^3 \right) + O(t^{-1}). \quad (3.20)$$

Furthermore, since $R_0 = -\frac{1}{2} \frac{1}{R_{-1}} \frac{dR_{-1}}{dt}$ has a singularity of the form $R_0 \sim -\frac{1}{8} (t - \tau_1)^{-1}$ at $t = \tau_1$, we obtain

$$\int_{\Gamma_t} R_0 dt = 4\pi i \operatorname{Res}_{t=\tau_1} R_0 = -\frac{\pi i}{2}.$$

This implies that

$$I_0(t, c, \eta) = 0. \quad (3.21)$$

Making use of (3.12), (3.19), (3.20) and (3.21) and taking the limit $t \rightarrow \infty$, we thus obtain the following.

Proposition 3.1. *The P -Voros coefficient $W(c, \eta)$ is a formal solution of the following difference equation:*

$$W(c, \eta) - W(c - \eta^{-1}, \eta) = -1 + c\eta \log \left(1 + \frac{1}{c\eta - 1} \right) - \log \left(1 + \frac{1}{2(c\eta - 1)} \right).$$

By Lemma 3.1 and Proposition 3.1, the proof of Theorem 3.1 is completed. \square

Using Theorem 3.1, we can explicitly analyze the parametric Stokes phenomenon for the P -Voros coefficient which occurs when $\arg c$ varies near $\frac{\pi}{2}$.

Corollary 3.1. *By denoting the Borel resummation operator by \mathcal{S} , we obtain the following:*

$$\mathcal{S}[e^{W(c, \eta)}]_{\arg c = \frac{\pi}{2} - \varepsilon} = (1 + e^{2\pi i c \eta}) \mathcal{S}[e^{W(c, \eta)}]_{\arg c = \frac{\pi}{2} + \varepsilon}, \quad (3.22)$$

where ε is a sufficiently small positive number.

Proof. By [T2, Theorem 1.1], the Voros coefficient $V_{\text{Weber}}(E, \eta)$ of the Weber equation (1.2) is represented as

$$V_{\text{Weber}}(E, \eta) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{2^{1-2n} - 1}{2n(2n-1)} B_{2n}(iE\eta)^{1-2n}. \quad (3.23)$$

Because the Voros coefficient of the Weber equation $V_{\text{Weber}}(E, \eta)$ and the P -Voros coefficient $W(c, \eta)$ are related as

$$W(c, \eta) = -2 V_{\text{Weber}}(-ic, \eta),$$

the relation (3.22) follows from [T2, Theorem 2.1] immediately. \square

3.3 Derivation of the connection formula for the parametric Stokes phenomena through the Voros coefficient of (P_{II})

We determine the connection formula for the parametric Stokes phenomenon which occurs when $\arg c$ varies near $\frac{\pi}{2}$ for the 1-parameter solutions λ_{∞} and λ_{τ_1} of (P_{II}) .

We know the following result about the Borel summability of WKB solutions of second order linear differential equations.

Theorem 3.2 ([DP, Theorem 1.2.2]). *Consider a second order linear differential equation of the form*

$$\left(\frac{d^2}{dx^2} - \eta^2 Q(x)\right)\psi = 0,$$

where $Q(x)$ is a polynomial of x . Let ψ_{\pm} be a WKB solution of the above equation normalized at infinity (like (3.3)), where the path of integration from $x = \infty$ is assumed to touch with no turning points and no Stokes curves. Then, ψ_{\pm} is Borel summable even under the situation where the degeneration of Stokes curves occurs.

This result suggests that no parametric Stokes phenomenon occurs (when $\arg c$ varies near $\frac{\pi}{2}$) for $\tilde{\lambda}_{\infty}^{(1)}$ given by (3.3), which is a WKB solution of (2.5) normalized at infinity along the path in Figure 3.2. That is, the following holds:

$$\begin{aligned} \mathcal{S}[\tilde{\lambda}_{\infty}^{(1)}(t, c, \eta; \alpha)|_{\arg c = \frac{\pi}{2} - \varepsilon}] &= \mathcal{S}[\tilde{\lambda}_{\infty}^{(1)}(t, c, \eta; \tilde{\alpha})|_{\arg c = \frac{\pi}{2} + \varepsilon}] \\ &\Rightarrow \tilde{\alpha} = \alpha. \end{aligned} \quad (3.24)$$

On the other hand, combining (3.5), (3.24) and Corollary 3.1, we obtain the following connection formula for $\tilde{\lambda}_{\tau_1}^{(1)}$ given by (3.1):

$$\begin{aligned} \mathcal{S}[\tilde{\lambda}_{\tau_1}^{(1)}(t, c, \eta; \alpha)|_{\arg c = \frac{\pi}{2} - \varepsilon}] &= \mathcal{S}[\tilde{\lambda}_{\tau_1}^{(1)}(t, c, \eta; \tilde{\alpha})|_{\arg c = \frac{\pi}{2} + \varepsilon}] \\ &\Rightarrow \alpha \mathcal{S}[e^W|_{\arg c = \frac{\pi}{2} - \varepsilon}] = \tilde{\alpha} \mathcal{S}[e^W|_{\arg c = \frac{\pi}{2} + \varepsilon}] \\ &\Rightarrow \tilde{\alpha} = (1 + e^{2\pi i c \eta}) \alpha. \end{aligned} \quad (3.25)$$

As noted in Section 2, 1-parameter solutions are determined uniquely once the normalizations of $\tilde{\lambda}^{(1)}$ are fixed. These observations suggest that the parametric Stokes phenomena for the 1-parameter solutions of (P_{II}) can be described as follows:

Connection formula for the 1-parameter solutions of (P_{II}) . *Let ε be a sufficiently small positive number.*

(i) *If the true solutions represented by $\lambda_{\infty}(t, c, \eta; \alpha)$ for $\arg c = \frac{\pi}{2} - \varepsilon$ and those by $\lambda_{\infty}(t, c, \eta; \tilde{\alpha})$ for $\arg c = \frac{\pi}{2} + \varepsilon$ coincide, then the following holds:*

$$\tilde{\alpha} = \alpha. \quad (3.26)$$

(ii) *If the true solutions represented by $\lambda_{\tau_1}(t, c, \eta; \alpha)$ for $\arg c = \frac{\pi}{2} - \varepsilon$ and those by $\lambda_{\tau_1}(t, c, \eta; \tilde{\alpha})$ for $\arg c = \frac{\pi}{2} + \varepsilon$ coincide, then the following holds:*

$$\tilde{\alpha} = (1 + e^{2\pi i c \eta}) \alpha. \quad (3.27)$$

In the subsequent sections, employing the exact WKB analysis for the Schrödinger equation (SL_{II}) associated with (P_{II}) through the isomonodromic deformation, we will rederive this connection formula in a completely different manner.

4 WKB solutions and the Stokes geometry of (SL_{II}) and (D_{II})

(P_{II}) represents the condition for isomonodromic deformation of the following associated Schrödinger equation (SL_{II}) :

$$(SL_{\text{II}}) : \left(\frac{\partial^2}{\partial x^2} - \eta^2 Q_{\text{II}}\right)\psi = 0,$$

where

$$Q_{\text{II}} = x^4 + tx^2 + 2cx + 2K_{\text{II}} - \eta^{-1} \frac{\nu}{x - \lambda} + \eta^{-2} \frac{3}{4(x - \lambda)^2},$$

$$K_{\text{II}} = \frac{1}{2}[\nu^2 - (\lambda^4 + t\lambda^2 + 2c\lambda)].$$

Here the regular singular point $x = \lambda$ of (SL_{II}) is an apparent singular point because K_{II} has the above form. We will construct WKB solutions of (SL_{II}) satisfying its deformation equation (D_{II}) in this section. Here the deformation equation (D_{II}) of (SL_{II}) is given by the following:

$$(D_{\text{II}}) : \frac{\partial \psi}{\partial t} = A_{\text{II}} \frac{\partial \psi}{\partial x} - \frac{1}{2} \frac{\partial A_{\text{II}}}{\partial x} \psi,$$

$$A_{\text{II}} = \frac{1}{2(x - \lambda)}.$$

Note that the system (SL_{II}) and (D_{II}) is obtained from the isomonodromic deformation equation of Jimbo-Miwa's form [JM, Appendix C].

Before discussing the construction of WKB solution satisfying both (SL_{II}) and (D_{II}) , we first establish the relation between degeneration of the P -Stokes curves of (P_{II}) and that of the Stokes curves of (SL_{II}) .

4.1 Stokes geometry of (SL_{II})

We investigate the Stokes geometry of (SL_{II}) in this subsection. By substituting a 1-parameter solution $(\lambda(t, c, \eta; \alpha), \nu(t, c, \eta; \alpha))$ of (H_{II}) into the coefficients of (SL_{II}) and (D_{II}) , we find that Q_{II} and A_{II} are expanded as follows:

$$Q_{\text{II}}(x, t, c, \eta; \alpha) = Q_{\text{II}}^{(0)}(x, t, c, \eta) + \alpha \eta^{-\frac{1}{2}} Q_{\text{II}}^{(1)}(x, t, c, \eta) e^{\eta \phi_{\text{II}}} + (\alpha \eta^{-\frac{1}{2}})^2 Q_{\text{II}}^{(2)}(x, t, c, \eta) e^{2\eta \phi_{\text{II}}} + \dots, \quad (4.1)$$

$$A_{\text{II}}(x, t, c, \eta; \alpha) = A_{\text{II}}^{(0)}(x, t, c, \eta) + \alpha \eta^{-\frac{1}{2}} A_{\text{II}}^{(1)}(x, t, c, \eta) e^{\eta \phi_{\text{II}}} + (\alpha \eta^{-\frac{1}{2}})^2 A_{\text{II}}^{(2)}(x, t, c, \eta) e^{2\eta \phi_{\text{II}}} + \dots, \quad (4.2)$$

$$Q_{\text{II}}^{(k)}(x, t, c, \eta) = Q_0^{(k)}(x, t, c) + \eta^{-1} Q_1^{(k)}(x, t, c) + \eta^{-2} Q_2^{(k)}(x, t, c) + \dots,$$

$$A_{\text{II}}^{(k)}(x, t, c, \eta) = A_0^{(k)}(x, t, c) + \eta^{-1} A_1^{(k)}(x, t, c) + \eta^{-2} A_2^{(k)}(x, t, c) + \dots.$$

Especially, $Q_{\text{II}}^{(0)}$ and $A_{\text{II}}^{(0)}$ and their leading terms are given by the following:

$$Q_{\text{II}}^{(0)}(x, t, c, \eta) = x^4 + tx^2 + 2cx + 2K_{\text{II}} - \eta^{-1} \frac{\nu^{(0)}}{x - \lambda^{(0)}} + \eta^{-2} \frac{3}{4(x - \lambda^{(0)})^2}, \quad (4.3)$$

$$A_{\text{II}}^{(0)}(x, t, c, \eta) = \frac{1}{2(x - \lambda^{(0)})}, \quad (4.4)$$

$$Q_0^{(0)}(x, t, c) = x^4 + tx^2 + 2cx - (\lambda_0^4 + t\lambda_0^2 + 2c\lambda_0)$$

$$= (x - \lambda_0)^2(x^2 + 2\lambda_0 x + 3\lambda_0^2 + t), \quad (4.5)$$

$$A_0^{(0)}(x, t, c) = \frac{1}{2(x - \lambda_0)}. \quad (4.6)$$

Here $(\lambda^{(0)}, \nu^{(0)})$ is the 0-parameter solution which is the principal part of the 1-parameter solution substituted. In what follows we abbreviate $Q_0^{(0)}(x, t, c)$ and $A_0^{(0)}(x, t, c)$ to $Q_0(x, t, c)$ and $A_0(x, t, c)$, respectively.

Definition 4.1 ([KT4, Definition 2.4, Definition 2.6]). (i) A point $x = a$ is called a *turning point of (SL_{II})* if x satisfies $Q_0(a, t, c) = 0$.
(ii) For a turning point $x = a$, a real one-dimensional curve defined by

$$\operatorname{Im} \int_a^x \sqrt{Q_0(x, t, c)} dx = 0$$

is called a *Stokes curve of (SL_{II})* .

In view of (4.5) we know that (SL_{II}) has a double turning point at $x = \lambda_0$ and two simple turning points which denoted by $x = a_1, a_2$ in what follows, where a_1 and a_2 are two roots of $x^2 + 2\lambda_0 x + 3\lambda_0^2 + t = 0$. Figure 4.1 ~ 4.3 describe the Stokes curves of (SL_{II}) with t being a fixed point in the region in Figure 2.3 and $\arg c$ varying near $\frac{\pi}{2}$. (Here ε is a sufficiently small positive number.) It is observed that a degeneration of the

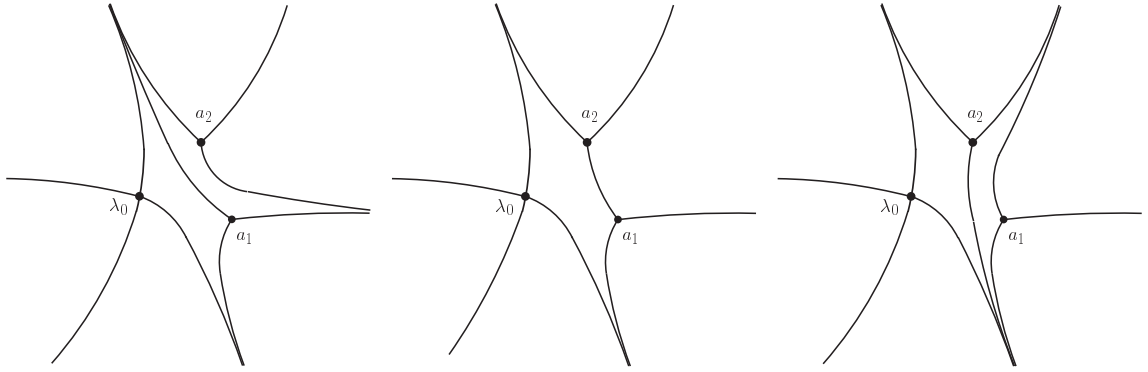


Figure 4.1: Stokes curve of (SL_{II}) when $\arg c = \frac{\pi}{2} - \varepsilon$. Figure 4.2: Stokes curve of (SL_{II}) when $\arg c = \frac{\pi}{2}$. Figure 4.3: Stokes curve of (SL_{II}) when $\arg c = \frac{\pi}{2} + \varepsilon$.

P -Stokes geometry of (P_{II}) and that of the Stokes geometry of (SL_{II}) occurs at $\arg c = \frac{\pi}{2}$ simultaneously. This intriguing observation can be confirmed analytically by the following proposition.

Proposition 4.1.

$$\int_{a_1}^{a_2} \sqrt{Q_0(x, t, c)} dx = -\frac{1}{2} \int_{\tau_1}^{\tau_2} \sqrt{\Delta(t, c)} dt = \pi i c. \quad (4.7)$$

Here the integral $\int_{a_1}^{a_2} \sqrt{Q_0(x, t, c)} dx$ is defined by $\frac{1}{2} \int_{\gamma} \sqrt{Q_0(x, t, c)} dx$, where γ designates the closed curve in the cut plane shown in Figure 4.4, and the path of the integral $\int_{\tau_1}^{\tau_2} \sqrt{\Delta(t, c)} dt$ is taken to be along the P -Stokes curve which connects two P -turning points $t = \tau_1$ and τ_2 in Figure 2.3. (The Wiggly line in Figure 4.4 is a cut to define the Riemann surface of $\sqrt{Q_0}$. We adopt the branch of $\sqrt{Q_0}$ such that $\sqrt{Q_0} \sim x^2$ as $x \rightarrow \infty$ in this cut plane.)

Proof. Firstly, we compute the integral $\int_{a_1}^{a_2} \sqrt{Q_0(x, t, c)} dx$. Because $\sqrt{Q_0}$ has no singularities besides at $x = a_1, a_2$ in \mathbb{C} , we have

$$\int_{\gamma} \sqrt{Q_0(x, t, c)} dx = -2\pi i \operatorname{Res}_{x=\infty} (\sqrt{Q_0(x, t, c)} dx). \quad (4.8)$$

We also obtain

$$\operatorname{Res}_{x=\infty} (\sqrt{Q_0(x, t, c)} dx) = -c$$

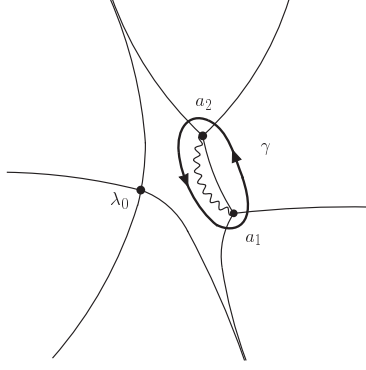


Figure 4.4: Integration path γ .

by (4.21) below. Hence we find

$$\int_{a_1}^{a_2} \sqrt{Q_0(x, t, c)} dx = \pi i c.$$

Next, we prove the first equality of (4.7). It follows from [KT4, Theorem 4.9] that, when t tends to a simple P -turning point, a simple turning point of (SL_{II}) merges with the double turning point $x = \lambda_0$. Under the current situation we can check that $x = a_j$ merges with the double turning point $x = \lambda_0$ at $t = \tau_j$ for $j = 1, 2$ respectively by numerical computation. Thus, by the same theorem in [KT4], we also have

$$\int_{a_j}^{\lambda_0} \sqrt{Q_0(x, t, c)} dx = \pm \frac{1}{2} \int_{\tau_j}^t \sqrt{\Delta(t, c)} dt \quad (j = 1, 2). \quad (4.9)$$

The sign of the right-hand side of (4.9) depends on the choice of the branch of $\sqrt{\Delta}$. Thus we obtain

$$\pi i c = \int_{a_1}^{a_2} \sqrt{Q_0(x, t, c)} dx = \mp \frac{1}{2} \int_{\tau_1}^{\tau_2} \sqrt{\Delta(t, c)} dt. \quad (4.10)$$

Since the branch of $\sqrt{\Delta}$ was determined such that the real part of $\int_{\tau_1}^{\tau_2} \sqrt{\Delta} dt$ is positive when $\arg c = \frac{\pi}{2}$ (see Figure 2.3), the sign of the right-hand side of (4.10) is $-$ (hence the sign of (4.9) is $+$), which completes the proof of Proposition 4.1. \square

4.2 WKB solutions of (SL_{II}) and (D_{II})

We construct WKB solutions of (SL_{II}) satisfying (D_{II}) simultaneously in this subsection. Although Q_{II} is expanded as (4.1), we can construct WKB solutions in the following form, similarly to [KT4, §2] (the well-definedness of the integral will be confirmed in Lemma 4.1 below):

$$\psi_{\pm, \infty} = \frac{1}{\sqrt{S_{\text{odd}}}} \exp \pm \left\{ \eta \int_{a_1}^x S_{-1} dx + \int_{\infty}^x (S_{\text{odd}} - \eta S_{-1}) dx \right\}, \quad (4.11)$$

where

$$S_{\text{odd}}(x, t, c, \eta; \alpha) = S_{\text{odd}}^{(0)}(x, t, c, \eta) + \alpha \eta^{-\frac{1}{2}} S_{\text{odd}}^{(1)}(x, t, c, \eta) e^{\eta \phi_{\text{II}}} + (\alpha \eta^{-\frac{1}{2}})^2 S_{\text{odd}}^{(2)}(x, t, c, \eta) e^{2\eta \phi_{\text{II}}} + \dots \quad (4.12)$$

is the odd part (in the sense of Remark 2.1) of a formal solution

$$S(x, t, c, \eta; \alpha) = S^{(0)}(x, t, c, \eta) + \alpha \eta^{-\frac{1}{2}} S^{(1)}(x, t, c, \eta) e^{\eta \phi_{\text{II}}} + (\alpha \eta^{-\frac{1}{2}})^2 S^{(2)}(x, t, c, \eta) e^{2\eta \phi_{\text{II}}} + \dots \quad (4.13)$$

of the associated Riccati equation of (SL_{II})

$$S^2 + \frac{\partial S}{\partial x} = \eta^2 Q_{\text{II}}(x, t, c, \eta; \alpha), \quad (4.14)$$

$S_{\text{odd}}^{(k)}$ and $S^{(k)}$ are formal power series of η^{-1} of the form

$$S_{\text{odd}}^{(k)}(x, t, c, \eta) = \eta S_{\text{odd}, -1}^{(k)}(x, t, c) + S_{\text{odd}, 0}^{(k)}(x, t, c) + \eta^{-1} S_{\text{odd}, 1}^{(k)}(x, t, c) + \cdots \quad (k \geq 0),$$

$$S^{(k)}(x, t, c, \eta) = \eta S_{-1}^{(k)}(x, t, c) + S_0^{(k)}(x, t, c) + \eta^{-1} S_1^{(k)}(x, t, c) + \cdots \quad (k \geq 0),$$

and

$$\begin{aligned} S_{-1}(x, t, c) &= S_{-1}^{(0)}(x, t, c) = \sqrt{Q_0(x, t, c)} \\ &= (x - \lambda_0) \sqrt{x^2 + 2\lambda_0 x + 3\lambda_0^2 + t}. \end{aligned} \quad (4.15)$$

The formal series $S^{(k)}$ ($k \geq 0$) satisfy the following differential equations:

$$S^{(0)2} + \frac{\partial S^{(0)}}{\partial x} = \eta^2 Q^{(0)}. \quad (4.16)$$

$$2S^{(0)}S^{(k)} + \sum_{k_1+k_2=k, k_j < k} S^{(k_1)}S^{(k_2)} + \frac{\partial S^{(k)}}{\partial x} = \eta^2 Q_{\text{II}}^{(k)} \quad (k \geq 1). \quad (4.17)$$

Once we fix the branch of S_{-1} , then $S_{\ell}^{(k)}$ is determined uniquely by the following recursive relations:

$$2S_{-1}S_{\ell+1}^{(k)} + \sum_{\substack{k_1+k_2=k \\ \ell_1+\ell_2=\ell \\ 0 \leq k_j < k, \ell_j \geq 0}} S_{\ell_1}^{(k_1)}S_{\ell_2}^{(k_2)} + \frac{\partial S_{\ell}^{(k)}}{\partial x} = Q_{\ell+2}^{(k)} \quad (k \geq 0, \ell \geq -2) \quad (4.18)$$

In what follows we adopt the branch of S_{-1} such that $S_{-1} \sim x^2$ as $x \rightarrow \infty$ on the cut plane shown in Figure 4.4.

Proposition 4.2.

$$S_{\text{odd}, -1}^{(k)} = S_{-1}^{(k)} = 0 \quad (k \geq 1). \quad (4.19)$$

Proof. The following relation is shown in [AKT1, Proposition 2.1]:

$$\frac{\partial}{\partial t} S = \frac{\partial}{\partial x} \left(A_{\text{II}} S - \frac{1}{2} \frac{\partial A_{\text{II}}}{\partial x} \right). \quad (4.20)$$

Comparing the coefficients of $e^{k\eta\phi_{\text{II}}}$ of both sides of the equation (4.20), we obtain

$$k\eta \frac{d\phi_{\text{II}}}{dt} S^{(k)} + \frac{\partial}{\partial t} S^{(k)} = \frac{\partial}{\partial x} \left(A_{\text{II}}^{(k)} S^{(0)} + \sum_{j=1}^k A^{(k-j)} S^{(j)} - \frac{1}{2} \frac{\partial A_{\text{II}}^{(k)}}{\partial x} \right).$$

By taking the coefficients of η^2 of both sides of this equation, we have $k \frac{d\phi_{\text{II}}}{dt} S_{-1}^{(k)} = 0$. Thus we obtain (4.19). It is obvious that $S_{\text{odd}, -1}^{(k)} = S_{-1}^{(k)}$ for all $k \geq 0$ by the definition. \square

We can check the following facts easily by straightforward computations.

Lemma 4.1. *The following asymptotic behaviors hold as $x \rightarrow \infty$:*

$$S_{-1}(x, t, c) = x^2 + \frac{t}{2} + \frac{c}{x} + O(x^{-2}), \quad (4.21)$$

$$S_0^{(0)}(x, t, c) = -x^{-1} + O(x^{-2}), \quad (4.22)$$

$$S_\ell^{(0)}(x, t, c) = O(x^{-2}) \quad (\ell \geq 1), \quad (4.23)$$

$$S_\ell^{(k)}(x, t, c) = O(x^{-2}) \quad (k \geq 1, \ell \geq 0). \quad (4.24)$$

Therefore, the integral in (4.11) is well-defined. We also note that S_{even} , the even part of S (in the sense of Remark 2.1), satisfies that

$$S_{\text{even}} = -\frac{1}{2} \frac{\partial}{\partial x} \log S_{\text{odd}}. \quad (4.25)$$

The WKB solutions $\psi_{\pm, \infty}$ do not satisfy (D_{II}) . As a matter of fact, we can verify the following proposition.

Proposition 4.3.

$$\frac{\partial}{\partial t} \psi_{\pm, \infty} = A_{\text{II}} \frac{\partial \psi_{\pm, \infty}}{\partial x} - \frac{1}{2} \frac{\partial A_{\text{II}}}{\partial x} \psi_{\pm, \infty} \mp \frac{1}{2} \eta (\lambda - \lambda_0) \psi_{\pm, \infty}. \quad (4.26)$$

Proof. Using (4.20), we have

$$\frac{\partial}{\partial t} S_{\text{odd}} = \frac{\partial}{\partial x} (A_{\text{II}} S_{\text{odd}}), \quad (4.27)$$

$$\frac{\partial}{\partial t} S_{-1} = \frac{\partial}{\partial x} (A_0 S_{-1}). \quad (4.28)$$

Taking these relations into account and differentiating $\psi_{\pm, \infty}$ with respect to t , we obtain the following:

$$\begin{aligned} \frac{\partial}{\partial t} \psi_{\pm, \infty} &= -\frac{1}{2} \frac{1}{S_{\text{odd}}} \frac{\partial S_{\text{odd}}}{\partial t} \psi_{\pm, \infty} \pm \left\{ \eta \int_{a_1}^x \frac{\partial S_{-1}}{\partial t} dx + \int_{\infty}^x \frac{\partial}{\partial t} (S_{\text{odd}} - \eta S_{-1}) \right\} \psi_{\pm, \infty} \\ &= -\frac{1}{2} \frac{1}{S_{\text{odd}}} \frac{\partial}{\partial x} (A_{\text{II}} S_{\text{odd}}) \psi_{\pm, \infty} \\ &\quad \pm \left\{ \eta \int_{a_1}^x \frac{\partial}{\partial x} (A_0 S_{-1}) dx + \int_{\infty}^x \frac{\partial}{\partial x} (A_{\text{II}} S_{\text{odd}} - \eta A_0 S_{-1}) \right\} \psi_{\pm, \infty} \\ &= -\frac{1}{2} \left(\frac{\partial A_{\text{II}}}{\partial x} + A_{\text{II}} \frac{1}{S_{\text{odd}}} \frac{\partial S_{\text{odd}}}{\partial x} \right) \psi_{\pm, \infty} \\ &\quad \pm A_{\text{II}} S_{\text{odd}} \psi_{\pm, \infty} \mp [A_{\text{II}} S_{\text{odd}} - \eta A_0 S_{-1}] \Big|_{x=\infty} \psi_{\pm, \infty}. \end{aligned} \quad (4.29)$$

Since A_{II} behaves as

$$A_{\text{II}} = \frac{1}{2(x - \lambda_0)} + \frac{\lambda - \lambda_0}{2(x - \lambda_0)^2} + O(x^{-3})$$

when x tends to ∞ , we have

$$[A_{\text{II}} S_{\text{odd}} - \eta A_0 S_{-1}] \Big|_{x=\infty} = \frac{1}{2} \eta (\lambda - \lambda_0) \quad (4.30)$$

in view of (4.21) \sim (4.24). Furthermore, it follows from the definition (4.11) of $\psi_{\pm, \infty}$ that

$$\frac{\partial \psi_{\pm, \infty}}{\partial x} = \left(\pm S_{\text{odd}} - \frac{1}{2} \frac{1}{S_{\text{odd}}} \frac{\partial S_{\text{odd}}}{\partial x} \right) \psi_{\pm, \infty}. \quad (4.31)$$

Making use of (4.29), (4.30) and (4.31), we have (4.26). \square

Proposition 4.3 implies that the WKB solutions

$$\psi_{\pm, \text{IM}} = e^{\pm \frac{1}{2}U} \psi_{\pm, \infty}$$

satisfy both (SL_{II}) and (D_{II}) , where

$$U = U(t, c, \eta; \alpha) = \eta \int_{\infty}^t (\lambda(t, c, \eta; \alpha) - \lambda_0(t, c)) dt. \quad (4.32)$$

(IM stands for Iso-Monodromic.) The Stokes multipliers of (SL_{II}) around $x = \infty$ will be computed in Section 6.3 by using $\psi_{\pm, \text{IM}}$.

Remark 4.1. The integral (4.32) is defined in the following sense:

$$U(t, c, \eta; \alpha) = U^{(0)}(t, c, \eta) + \alpha \eta^{-\frac{1}{2}} U^{(1)}(t, c, \eta) e^{\eta \phi_{\text{II}}} + (\alpha \eta^{-\frac{1}{2}})^2 U^{(2)}(t, c, \eta) e^{2\eta \phi_{\text{II}}} + \dots, \quad (4.33)$$

$$U^{(k)}(t, c, \eta) = U_0^{(k)}(t, c) + \eta^{-1} U_1^{(k)}(t, c) + \eta^{-2} U_2^{(k)}(t, c) + \dots \quad (k \geq 0),$$

where $U^{(0)}$ is defined by the following integral whose integration path is the same as the normalization path of $\tilde{\lambda}_{\infty}^{(1)}$ (see Figure 3.2):

$$U^{(0)}(t, c, \eta) = \eta \int_{\infty}^t (\lambda^{(0)}(t, c, \eta) - \lambda_0(t, c)) dt, \quad (4.34)$$

and $U^{(k)}$ ($k \geq 1$) is the unique formal series satisfying

$$k\eta \frac{d\phi_{\text{II}}}{dt} U^{(k)} + \frac{\partial}{\partial t} U^{(k)} = \frac{1}{2} \eta \lambda^{(k)}.$$

This is not an integral in the usual sense, but U satisfies

$$\frac{\partial}{\partial t} U = \eta (\lambda - \lambda_0). \quad (4.35)$$

The end point $t = \infty$ of the integral in (4.32) is chosen so as to obtain an explicit representation of the ‘‘Voros coefficient’’ of (SL_{II}) , which will be calculated in the next section. We also note that, making use of Lemma 4.1 and (5.20) below, we have the following asymptotic behavior of $\psi_{\pm, \text{IM}}$ as x tends to ∞ :

$$\psi_{\pm, \text{IM}} = \eta^{-\frac{1}{2}} x^{-1 \pm c\eta} \exp \pm \left\{ \eta \left(\frac{1}{3} x^3 + \frac{1}{2} t x \right) - \frac{1}{2} \left(\frac{4}{3} \eta \lambda_0^3 + c\eta \log \left(-\frac{2\lambda_0^2 + t}{4} \right) - U \right) \right\} (1 + O(x^{-1})). \quad (4.36)$$

If we define u by

$$\log u = \frac{4}{3} \eta \lambda_0^3 + c\eta \log \left(-\frac{2\lambda_0^2 + t}{4} \right) - U, \quad (4.37)$$

we can verify that

$$\frac{d}{dt} \log u = -\eta \lambda.$$

Hence, the quantity u defined above is nothing but u which appeared in [JM, (C.10)-(C.13) in Appendix].

Remark 4.2. Since S_{odd} and U are expanded as (4.12) and (4.33) respectively, $\psi_{\pm, \text{IM}}$ is expanded as follows (see also Proposition 4.2):

$$\psi_{\pm, \text{IM}}(x, t, c, \eta; \alpha) = \psi_{\pm}^{(0)}(x, t, c, \eta) + \alpha \eta^{-\frac{1}{2}} \psi_{\pm}^{(1)}(x, t, c, \eta) e^{\eta \phi_{\text{II}}} + (\alpha \eta^{-\frac{1}{2}})^2 \psi_{\pm}^{(2)}(x, t, c, \eta) e^{2\eta \phi_{\text{II}}} + \dots, \quad (4.38)$$

$$\psi_{\pm}^{(k)}(x, t, c, \eta) = \eta^{-\frac{1}{2}} \left\{ \psi_{\pm, 0}^{(k)}(x, t, c) + \eta^{-1} \psi_{\pm, 1}^{(k)}(x, t, c) + \eta^{-2} \psi_{\pm, 2}^{(k)}(x, t, c) + \dots \right\} \exp \pm \eta \left(\int_{a_1}^x S_{-1}(x, t, c) dx \right).$$

Especially, $\psi_{\pm}^{(0)}$ is given by the following:

$$\psi_{\pm}^{(0)}(x, t, c, \eta) = \frac{1}{\sqrt{S_{\text{odd}}^{(0)}}} \exp \pm \left\{ \frac{1}{2} U^{(0)} + \int_{\infty}^x (S_{\text{odd}}^{(0)} - \eta S_{-1}) dx \right\} \exp \pm \eta \left(\int_{a_1}^x S_{-1} dx \right). \quad (4.39)$$

Remark 4.3. Because $\text{Res}_{x=\infty} \{ (S_{\text{odd}} - \eta S_{-1}) dx \} = 0$ by (4.23) and (4.24), we have

$$\int_{a_1}^{a_2} S_{\text{odd}} dx = \pi i c \eta. \quad (4.40)$$

(The left-hand side is defined by the contour integral $\frac{1}{2} \int_{\gamma} S_{\text{odd}} dx$, where γ is a closed path in Figure 4.4.) This relation will be used in the computations of the Stokes multipliers of (SL_{II}) .

5 The Voros coefficient of (SL_{II})

In this section, as a preparation of the computation of the Stokes multipliers of (SL_{II}) , we investigate the Voros coefficient of (SL_{II}) .

5.1 The Voros coefficient of (SL_{II})

First we define the Voros coefficient of (SL_{II}) .

Definition 5.1. The Voros coefficient of (SL_{II}) is given by

$$V(t, c, \eta; \alpha) = \int_{a_1}^{\infty} (S_{\text{odd}}(x, t, c, \eta; \alpha) - \eta S_{-1}(x, t, c)) dx. \quad (5.1)$$

Here the right-hand side of (5.1) is defined by

$$\frac{1}{2} \int_{\gamma_{\infty}} (S_{\text{odd}}(x, t, c, \eta; \alpha) - \eta S_{-1}(x, t, c)) dx,$$

where γ_{∞} is a path on the Riemann surface of $\sqrt{Q_0}$ shown in Figure 5.1. (The dotted part of γ_{∞} represents a path on the other sheet of the Riemann surface of $\sqrt{Q_0}$.) The right-hand side of (5.1) is well-defined by Lemma 4.1.

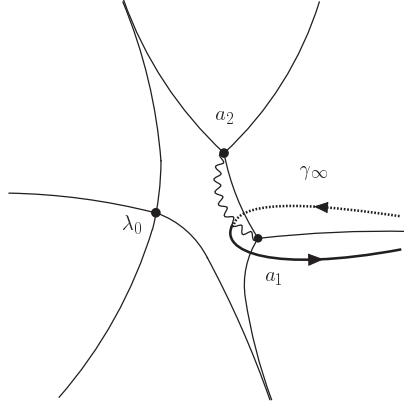


Figure 5.1: Integration path γ_{∞} .

Because S_{odd} is expanded as (4.13), V is also expanded as follows:

$$V(t, c, \eta; \alpha) = V^{(0)}(t, c, \eta) + \alpha \eta^{-\frac{1}{2}} V^{(1)}(t, c, \eta) e^{\eta \phi_{\text{II}}} + (\alpha \eta^{-\frac{1}{2}})^2 V^{(2)}(t, c, \eta) e^{2\eta \phi_{\text{II}}} + \dots, \quad (5.2)$$

$$V^{(0)}(t, c, \eta) = \int_{a_1}^{\infty} (S_{\text{odd}}^{(0)}(x, t, c, \eta) - \eta S_{-1}(x, t, c)) dx = V_0^{(0)}(t, c) + \eta^{-1} V_1^{(0)}(t, c) + \dots, \quad (5.3)$$

$$V^{(k)}(t, c, \eta) = \int_{a_1}^{\infty} S_{\text{odd}}^{(k)}(x, t, c, \eta) dx = V_0^{(k)}(t, c) + \eta^{-1} V_1^{(k)}(t, c) + \dots \quad (k \geq 1).$$

Proposition 5.1.

$$\frac{\partial}{\partial t}V = \frac{1}{2}\eta(\lambda - \lambda_0). \quad (5.4)$$

Proof. Using (4.27), (4.28) and (4.30), we can compute $\frac{\partial}{\partial t}V$ as follows: we have

$$\begin{aligned} \frac{\partial}{\partial t}V &= \frac{1}{2} \int_{\gamma_\infty} \frac{\partial}{\partial t}(S_{\text{odd}} - \eta S_{-1})dx \\ &= \frac{1}{2} \int_{\gamma_\infty} \frac{\partial}{\partial x}(A_{\text{II}}S_{\text{odd}} - \eta A_0 S_{-1})dx \\ &= \frac{1}{2}\eta(\lambda - \lambda_0). \end{aligned}$$

□

Proposition 5.1 together with (4.35) shows that the quantity $2V - U$ is independent of t . Since $2V - U$ is expanded as

$$2V - U = (2V^{(0)} - U^{(0)}) + \alpha\eta^{-\frac{1}{2}}(2V^{(1)} - U^{(1)})e^{\eta\phi_{\text{II}}} + (\alpha\eta^{-\frac{1}{2}})^2(2V^{(2)} - U^{(2)})e^{2\eta\phi_{\text{II}}} + \dots,$$

this independence of t implies that all the coefficients of $e^{k\eta\phi_{\text{II}}}$ ($k \geq 1$) must vanish. So we obtain the following:

Proposition 5.2.

$$2V(t, c, \alpha, \eta) - U(t, c, \alpha, \eta) = 2V^{(0)}(t, c, \eta) - U^{(0)}(t, c, \eta) \quad (5.5)$$

and this is independent of t .

In view of (4.3), (4.16), and (5.3) $V^{(0)}$ coincides with the Voros coefficient of (SL_{II}) with a 0-parameter solution of (H_{II}) substituted into the coefficients of Q_{II} . Hence, to compute $2V - U$, it suffices to consider (SL_{II}) with a 0-parameter solution substituted.

5.2 Determination of the Voros coefficient of (SL_{II})

We have an explicit description of the quantity $2V - U$.

Theorem 5.1. $2V - U$ is represented explicitly as follows:

$$2V(t, c, \eta) - U(t, c, \eta) = - \sum_{n=1}^{\infty} \frac{2^{1-2n} - 1}{2n(2n-1)} B_{2n}(c\eta)^{1-2n}, \quad (5.6)$$

where B_{2n} is the $2n$ -th Bernoulli number defined by (1.4).

We will prove Theorem 5.1 in the remainder of this section. The idea of proof is similar to that of Theorem 3.1, that is, we derive the difference equation for $V^{(0)}$, which is the Voros coefficient of (SL_{II}) with a 0-parameter solution substituted.

Lemma 5.1.

$$V^{(0)}(t, c, \eta) = \frac{1}{2} \int_{\gamma_\infty} (S^{(0)}(x, t, c, \eta) - \eta S_{-1}(x, t, c) - S_0^{(0)}(x, t, c))dx. \quad (5.7)$$

We omit the proof of Lemma 5.1 because we can show it similarly to Lemma 3.2. By Lemma 5.1, if we define

$$\begin{aligned} J(x, t, c, \eta) &= \int_{\gamma_x} S^{(0)}(x, t, c, \eta)dx - \int_{\gamma_x} S^{(0)}(x, t, c - \eta^{-1}, \eta) dx, \\ J_j(x, t, c, \eta) &= \int_{\gamma_x} S_j^{(0)}(x, t, c)dx - \int_{\gamma_x} S_j^{(0)}(x, t, c - \eta^{-1}) dx \quad (j \geq -1), \end{aligned}$$

then we obtain

$$2V^{(0)}(t, c, \eta) - 2V^{(0)}(t, c - \eta^{-1}, \eta) = \lim_{x \rightarrow \infty} (J(x, t, c, \eta) - \eta J_{-1}(x, t, c, \eta) - J_0(x, t, c, \eta)). \quad (5.8)$$

Here γ_x is a path on the Riemann surface of $\sqrt{Q_0}$ shown in Figure 5.2, where \tilde{x} represents a point on the Riemann surface of $\sqrt{Q_0}$ satisfying $Q_0(x, t, c) = Q_0(\tilde{x}, t, c)$ and $\sqrt{Q_0(x, t, c)} = -\sqrt{Q_0(\tilde{x}, t, c)}$. (The dotted part of γ_x represents a path on the other sheet of the Riemann surface of $\sqrt{Q_0}$.) To calculate the right-hand side of (5.8), we employ

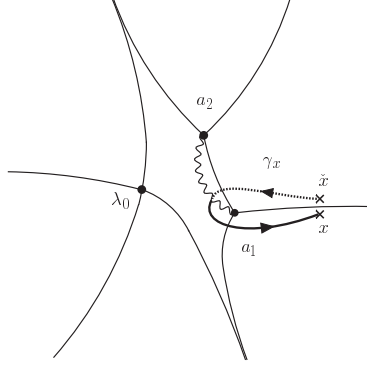


Figure 5.2: Integration path γ_x .

the so-called Schlesinger transformation of (SL_{II}) , which induces the translation of the parameter $c \mapsto c - \eta^{-1}$, as the counterpart of Bäcklund transformation of (P_{II}) in Lemma 3.3.

Lemma 5.2 ([JM, pp.423-424]). *Let $\psi = \psi(x, t, c, \eta)$ be a solution of (SL_{II}) with a solution (λ, ν) of (H_{II}) substituted into Q_{II} . If we define $\varphi = \varphi(x, t, c, \eta)$ by*

$$\varphi(x, t, c, \eta) = f(x, t, c, \eta)\psi(x, t, c, \eta) + g(x, t, c, \eta)\frac{\partial \psi}{\partial x}(x, t, c, \eta) \quad (5.9)$$

with

$$\begin{aligned} f &= (x - \Lambda(\lambda, \nu))^{-\frac{1}{2}}(x - \lambda)^{-\frac{1}{2}} \left\{ x^2 - \lambda^2 + \nu - \eta^{-1} \frac{1}{2(x - \lambda)} \right\}, \\ g &= -\eta^{-1} (x - \Lambda(\lambda, \nu))^{-\frac{1}{2}}(x - \lambda)^{-\frac{1}{2}}, \end{aligned}$$

then φ satisfies the following:

$$\left(\frac{\partial^2}{\partial x^2} - \eta^2 \hat{Q}_{\text{II}} \right) \varphi = 0, \quad (5.10)$$

where

$$\hat{Q}_{\text{II}} = x^4 + tx^2 + 2(c - \eta^{-1})x + 2\hat{K}_{\text{II}} - \eta^{-1} \frac{\mathcal{N}(\lambda, \nu)}{x - \Lambda(\lambda, \nu)} + \eta^{-2} \frac{3}{4(x - \Lambda(\lambda, \nu))^2},$$

$$\hat{K}_{\text{II}} = \frac{1}{2} \left[\mathcal{N}(\lambda, \nu)^2 - (\Lambda(\lambda, \nu)^4 + t \Lambda(\lambda, \nu)^2 + 2(c - \eta^{-1})\Lambda(\lambda, \nu)) \right].$$

(See Lemma 3.3 for the definition of $(\Lambda(\lambda, \nu), \mathcal{N}(\lambda, \nu))$.)

Lemma 5.2 can be shown by straightforward computations. With the aid of this Schlesinger transformation, we can derive the difference equation for $S^{(0)}$.

Lemma 5.3. Let $S^{(0)}(x, t, c, \eta)$ be a formal power series solution of the Riccati equation (4.14) associated with (SL_{II}) with a 0-parameter solution $(\lambda^{(0)}(t, c, \eta), \nu^{(0)}(t, c, \eta))$ of (H_{II}) substituted into the coefficients of Q_{II} . Then $S^{(0)}(x, t, c, \eta)$ satisfies the following:

$$S^{(0)}(x, t, c, \eta) - S^{(0)}(x, t, c - \eta^{-1}, \eta) = -\frac{\partial}{\partial x} \log \left(f^{(0)}(x, t, c, \eta) + g^{(0)}(x, t, c, \eta) S^{(0)}(x, t, c, \eta) \right), \quad (5.11)$$

where $f^{(0)}(x, t, c, \eta)$ and $g^{(0)}(x, t, c, \eta)$ are the formal power series of η^{-1} obtained by substituting $(\lambda^{(0)}, \nu^{(0)})$ into the expressions of f and g in Lemma 5.2, respectively.

Proof. We apply the Schlesinger transformation of Lemma 5.2 to a WKB solution

$$\psi(x, t, c, \eta) = \exp \left(\int^x S^{(0)}(x, t, c, \eta) dx \right)$$

of (SL_{II}) with $(\lambda^{(0)}, \nu^{(0)})$ substituted into its coefficients. Then, by Lemma 5.2,

$$\begin{aligned} \varphi(x, t, c, \eta) &= f^{(0)}(x, t, c, \eta) \psi(x, t, c, \eta) + g^{(0)}(x, t, c, \eta) \frac{\partial \psi}{\partial x}(x, t, c, \eta) \\ &= \left(f^{(0)}(x, t, c, \eta) + g^{(0)}(x, t, c, \eta) S^{(0)}(x, t, c, \eta) \right) \psi(x, t, c, \eta) \end{aligned} \quad (5.12)$$

satisfies (5.10) with $(\lambda^{(0)}, \nu^{(0)})$ substituted into the coefficients of \hat{Q}_{II} . On the other hand, since

$$(\Lambda(\lambda^{(0)}, \nu^{(0)}), \mathcal{N}(\lambda^{(0)}, \nu^{(0)})) = (\lambda^{(0)}(t, c - \eta^{-1}, \eta), \nu^{(0)}(t, c - \eta^{-1}, \eta))$$

holds as noted in Lemma 3.4 (i), the potential of the differential equation satisfied by φ of (5.12) is given by $Q_{II}^{(0)}(x, t, c - \eta^{-1}, \eta)$. Therefore, φ is also represented as

$$\varphi = C(\eta) \exp \left(\int^x S^{(0)}(x, t, c - \eta^{-1}, \eta) dx \right) \quad (5.13)$$

for some formal power series $C(\eta)$ of η^{-1} whose coefficients are independent of x . Thus, taking the logarithmic derivatives with respect to x of (5.12) and (5.13), we have the required relation (5.11). \square

By Lemma 5.3, J is represented as

$$J(x, t, c, \eta) = -\log \left\{ \frac{f^{(0)}(x, t, c, \eta) + g^{(0)}(x, t, c, \eta) S^{(0)}(x, t, c, \eta)}{f^{(0)}(x, t, c, \eta) + g^{(0)}(x, t, c, \eta) S^{(0)}(\tilde{x}, t, c, \eta)} \right\}. \quad (5.14)$$

In order to know the asymptotic behavior of J when x tends to ∞ , we have to investigate the asymptotic behaviors of $f^{(0)}$, $g^{(0)}$ and $S^{(0)}$. The asymptotic behaviors of $f^{(0)}$ and $g^{(0)}$ are given by

$$\begin{aligned} f^{(0)}(x, t, c, \eta) &= x + \frac{1}{2}(\lambda^{(0)} + \Lambda(\lambda^{(0)}, \nu^{(0)})) \\ &\quad + \left(\nu^{(0)} - \frac{5}{8}\lambda^{(0)2} + \frac{1}{4}\lambda^{(0)}\Lambda(\lambda^{(0)}, \nu^{(0)}) + \frac{3}{8}\Lambda(\lambda^{(0)}, \nu^{(0)})^2 \right) x^{-1} \\ &\quad + O(x^{-2}), \end{aligned} \quad (5.15)$$

$$\begin{aligned} g^{(0)}(x, t, c, \eta) &= -\eta^{-1} \left\{ x^{-1} + \frac{1}{2}(\lambda^{(0)} + \Lambda(\lambda^{(0)}, \nu^{(0)})) x^{-2} \right. \\ &\quad \left. + \left(\frac{3}{8}\lambda^{(0)2} + \frac{1}{4}\lambda^{(0)}\Lambda(\lambda^{(0)}, \nu^{(0)}) + \frac{3}{8}\Lambda(\lambda^{(0)}, \nu^{(0)})^2 \right) x^{-3} \right\} \\ &\quad + O(x^{-4}). \end{aligned} \quad (5.16)$$

(Here we take the branch $(x - \Lambda(\lambda, \nu))^{-\frac{1}{2}}(x - \lambda)^{-\frac{1}{2}} \sim x^{-1}$ as x tends to ∞ . We note that the choice of the branch of $(x - \Lambda(\lambda, \nu))^{-\frac{1}{2}}(x - \lambda)^{-\frac{1}{2}}$ does not affect to (5.14).) The behavior of $S^{(0)}$ follows from Lemma 4.1:

$$S^{(0)}(x, t, c, \eta) = +\left(\eta x^2 + \frac{1}{2}t\eta + (c\eta - 1)x^{-1} + \mathcal{O}(x^{-2})\right), \quad (5.17)$$

$$S^{(0)}(\check{x}, t, c, \eta) = -\left(\eta x^2 + \frac{1}{2}t\eta + (c\eta + 1)x^{-1} + \mathcal{O}(x^{-2})\right). \quad (5.18)$$

Hence, we have the following asymptotic behavior of J :

$$J = -\log\left\{\frac{1}{2}(\nu^{(0)} - \lambda^{(0)})^2 - \frac{1}{2}t x^{-2}\right\} + \mathcal{O}(x^{-1}). \quad (5.19)$$

Because we can compute the integral of S_{-1} explicitly as

$$\begin{aligned} \int_{\gamma_x} S_{-1}(x, t, c) dx &= \frac{2}{3}(x^2 + 2\lambda_0 x + 3\lambda_0^2 + t)^{\frac{3}{2}} - 2\lambda_0(x + \lambda_0)(x^2 + 2\lambda_0 x + 3\lambda_0^2 + t)^{\frac{1}{2}} \\ &\quad + c \log\left\{\frac{2(x + \lambda_0) + 2(x^2 + 2\lambda_0 x + 3\lambda_0^2 + t)^{\frac{1}{2}}}{2(x + \lambda_0) - 2(x^2 + 2\lambda_0 x + 3\lambda_0^2 + t)^{\frac{1}{2}}}\right\} \\ &= \frac{2}{3}x^3 + tx + 2c \log x - \frac{4}{3}\lambda_0^3 - c \log\left(-\frac{2\lambda_0^2 + t}{4}\right) + \mathcal{O}(x^{-1}), \end{aligned} \quad (5.20)$$

we also have the following asymptotic behavior of J_{-1} :

$$\begin{aligned} \eta J_{-1} &= -\frac{4}{3}\eta(\lambda_0(t, c)^3 - \lambda_0(t, c - \eta^{-1})^3) \\ &\quad + c \log\left\{\frac{2\lambda_0(t, c - \eta^{-1})^2 + t}{2\lambda_0(t, c)^2 + t}\right\} \\ &\quad + \log\left\{-\frac{4x^2}{2\lambda_0(t, c - \eta^{-1})^2 + t}\right\} + \mathcal{O}(x^{-1}). \end{aligned} \quad (5.21)$$

Furthermore, since $S_0^{(0)} = -\frac{1}{2}\frac{1}{S_{-1}}\frac{\partial S_{-1}}{\partial x}$ is single valued at $x = a_1$ and has a singularity of the form $S_0^{(0)} \sim -\frac{1}{4}(x - a_1)^{-1}$ as x tends to a_1 , we have

$$\int_{\gamma_x} S_0^{(0)} dx = 2\pi i \operatorname{Res}_{x=a_1} S_0^{(0)} = -\frac{\pi i}{2}.$$

This implies that

$$J_0 = 0. \quad (5.22)$$

Making use of (5.19), (5.21) and (5.22), we have the following asymptotic behavior:

$$\begin{aligned} J - \eta J_{-1} - J_0 &= \frac{4}{3}\eta(\lambda_0(t, c)^3 - \lambda_0(t, c - \eta^{-1})^3) \\ &\quad - c\eta \log\left\{\frac{2\lambda_0(t, c - \eta^{-1})^2 + t}{2\lambda_0(t, c)^2 + t}\right\} \\ &\quad - \log\left\{\frac{2\lambda^{(0)}(t, c, \eta)^2 + t - 2\nu^{(0)}(t, c, \eta)}{2\lambda_0(t, c - \eta^{-1})^2 + t}\right\} + \mathcal{O}(x^{-1}). \end{aligned} \quad (5.23)$$

Therefore, by taking the limit $x \rightarrow \infty$ in both sides of (5.8), we obtain the difference equation for $V^{(0)}$ as follows:

Proposition 5.3. *The Voros coefficient $V^{(0)}(t, c, \eta)$ satisfies the following difference equation formally:*

$$\begin{aligned} 2V^{(0)}(t, c, \eta) - 2V^{(0)}(t, c - \eta^{-1}, \eta) &= \frac{4}{3}\eta(\lambda_0(t, c)^3 - \lambda_0(t, c - \eta^{-1})^3) \\ &\quad - c\eta \log \left\{ \frac{2\lambda_0(t, c - \eta^{-1})^2 + t}{2\lambda_0(t, c)^2 + t} \right\} \\ &\quad - \log \left\{ \frac{2\lambda^{(0)}(t, c, \eta)^2 + t - 2\nu^{(0)}(t, c, \eta)}{2\lambda_0(t, c - \eta^{-1})^2 + t} \right\}. \end{aligned} \quad (5.24)$$

Although it is difficult to solve (5.24) explicitly, we can obtain the following fact from this relation.

Proposition 5.4. *The limit $V^{(0)}(\infty, c, \eta)$ of $V^{(0)}(t, c, \eta)$ as t tends to ∞ along the P -Stokes curve Γ in Figure 2.3 is represented explicitly as follows:*

$$V^{(0)}(\infty, c, \eta) = -\frac{1}{2} \sum_{n=1}^{\infty} \frac{2^{1-2n} - 1}{2n(2n-1)} B_{2n}(c\eta)^{1-2n}. \quad (5.25)$$

Proof. Since $2V^{(0)} - U^{(0)}$ is independent of t and $U^{(0)} \rightarrow 0$ as $t \rightarrow \infty$, the limit

$$V^{(0)}(\infty, c, \eta) = V_0^{(0)}(\infty, c) + \eta^{-1}V_1^{(1)}(\infty, c) + \eta^{-2}V_2^{(2)}(\infty, c) + \dots$$

is well-defined. Moreover, because it follows from (A.23) in Appendix that $V^{(0)}$ is invariant under the scaling $(c, \eta) \mapsto (r^{-1}c, r\eta)$, $V^{(0)}(\infty, c, \eta)$ is written in the form $\sum_{n=1}^{\infty} v_n(c\eta)^{-n}$ where $v_n \in \mathbb{C}$ is independent of c . Hence it suffices to show that $2V^{(0)}(\infty, c, \eta)$ satisfies the difference equation (3.10) by Lemma 3.1. Because the following asymptotic behaviors for $t \rightarrow \infty$

$$\lambda_0(t, c)^3 - \lambda_0(t, c - \eta^{-1})^3 = -\frac{3}{4}\eta^{-1} + O(t^{-\frac{3}{2}}), \quad (5.26)$$

$$2\lambda_0(t, c)^2 + t = -\sqrt{2}ict^{-\frac{1}{2}} + O(t^{-2}), \quad (5.27)$$

$$2\lambda^{(0)}(t, c, \eta)^2 + t - 2\nu^{(0)}(t, c, \eta) = \frac{i}{\sqrt{2}}(1 - 2c\eta)\eta^{-1}t^{-\frac{1}{2}} + O(t^{-2}), \quad (5.28)$$

hold by Lemma 2.1, by taking the limit $t \rightarrow \infty$ in both sides of (5.24), we have

$$2V^{(0)}(\infty, c, \eta) - 2V^{(0)}(\infty, c - \eta^{-1}, \eta) = -1 + c\eta \log \left(1 + \frac{1}{c\eta - 1} \right) - \log \left(1 + \frac{1}{2(c\eta - 1)} \right).$$

Thus the proof of Proposition 5.4 is completed. \square

Since $2V - U = 2V^{(0)} - U^{(0)}$ is independent on t , we obtain

$$2V^{(0)}(t, c, \eta) - U^{(0)}(t, c, \eta) = 2V^{(0)}(\infty, c, \eta)$$

by taking the limit $t \rightarrow \infty$. (Note that $U^{(0)} \rightarrow 0$ as $t \rightarrow \infty$.) Hence Theorem 5.1 is derived from Proposition 5.2 and Proposition 5.4 immediately.

Corollary 5.1.

$$\mathcal{S}[e^{2V-U}|_{\arg c = \frac{\pi}{2} - \varepsilon}] = (1 + e^{2\pi i c \eta}) \mathcal{S}[e^{2V-U}|_{\arg c = \frac{\pi}{2} + \varepsilon}], \quad (5.29)$$

where ε is a sufficiently small positive number.

Proof. Because $2V - U = W$ holds, Corollary 5.1 follows from Corollary 3.1 immediately. \square

As we will see in Section 6, the t -independent quantity $2V - U$ appears in the expression of the Stokes multipliers of (SL_{Π}) . Corollary 5.1 will be used in the derivation of the connection formulas for the parametric Stokes phenomenon of 1-parameter solutions in Section 6.4.

6 The Stokes multipliers of (SL_{II}) around $x = \infty$ and the parametric Stokes phenomena of (P_{II})

To seek the connection formulas for $\psi_{\pm, \text{IM}}$ on Stokes curves of (SL_{II}) emanating from the double turning point $x = \lambda_0$, we make use of the local transformation theory established in [AKT1] (cf. [KT2], [KT3] and [T1] also), which reduces (SL_{II}) to the following canonical equation (Can) in a neighborhood of $x = \lambda_0$:

$$(Can) : \left(\frac{\partial^2}{\partial \tilde{x}^2} - \eta^2 Q_{\text{can}} \right) \tilde{\psi} = 0,$$

where

$$\begin{aligned} Q_{\text{can}} = Q_{\text{can}}(\tilde{x}, \tilde{E}, \tilde{\sigma}, \tilde{\rho}, \eta) &= 4\tilde{x}^2 + \eta^{-1} \tilde{E} + \frac{\eta^{-\frac{3}{2}} \tilde{\rho}}{\tilde{x} - \eta^{-\frac{1}{2}} \tilde{\sigma}} + \eta^{-2} \frac{3}{4(\tilde{x} - \eta^{-\frac{1}{2}} \tilde{\sigma})^2}, \\ \tilde{E} &= \tilde{\rho}^2 - 4\tilde{\sigma}^2. \end{aligned}$$

Note that one peculiar feature of (SL_{II}) is that the location of the double turning point $x = \lambda_0$ coincides with the leading term of the apparent singular point $x = \lambda$ of (SL_{II}) . The canonical equation (Can) shares the same structure.

Later we need to consider the deformation equation (D_{can}) of (Can) which is given by the following:

$$(D_{\text{can}}) : \frac{\partial \tilde{\psi}}{\partial \tilde{t}} = A_{\text{can}} \frac{\partial \tilde{\psi}}{\partial \tilde{x}} - \frac{1}{2} \frac{\partial A_{\text{can}}}{\partial \tilde{x}} \tilde{\psi},$$

$$A_{\text{can}} = \frac{1}{2(\tilde{x} - \eta^{-\frac{1}{2}} \tilde{\sigma})}.$$

The compatibility condition of the system (Can) and (D_{can}) is represented as the following Hamiltonian system (H_{can}) :

$$(H_{\text{can}}) \begin{cases} \frac{d\tilde{\sigma}}{d\tilde{t}} = -\eta \tilde{\rho}, \\ \frac{d\tilde{\rho}}{d\tilde{t}} = -4\eta \tilde{\sigma}. \end{cases}$$

A general solution of (H_{can}) is given by

$$\begin{cases} \tilde{\sigma}(\tilde{t}, A, B, \eta) = A e^{2\eta \tilde{t}} + B e^{-2\eta \tilde{t}}, \\ \tilde{\rho}(\tilde{t}, A, B, \eta) = -2A e^{2\eta \tilde{t}} + 2B e^{-2\eta \tilde{t}}, \end{cases} \quad (6.1)$$

where A and B are free parameters. Taking appropriate parameters A and B , we will obtain a full order correspondence between the WKB solutions $\psi_{\pm, \text{IM}}$ of (SL_{II}) and (D_{II}) and some WKB solutions of (Can) and (D_{can}) through the transformation theory.

6.1 Transformation of (SL_{II}) near the double turning point $x = \lambda_0$

First we review the transformation theory from (SL_{II}) to (Can) .

Theorem 6.1 ([AKT1, Theorem3.1], [KT3, Theorem2.1]). *There exist a neighborhood \mathcal{U} of $x = \lambda_0$ and a formal series*

$$\begin{aligned} x_{\text{II}} &= x_{\text{II}}(x, t, c, \eta; \alpha) \\ &= x_{\text{II}}^{(0)}(x, t, c, \eta) + \alpha \eta^{-\frac{1}{2}} x_{\text{II}}^{(1)}(x, t, c, \eta) e^{\eta \phi_{\text{II}}} + (\alpha \eta^{-\frac{1}{2}})^2 x_{\text{II}}^{(2)}(x, t, c, \eta) e^{2\eta \phi_{\text{II}}} + \cdots, \end{aligned} \quad (6.2)$$

$$x_{\text{II}}^{(k)}(x, t, c, \eta) = x_0^{(k)}(x, t, c) + \eta^{-1} x_1^{(k)}(x, t, c) + \eta^{-2} x_2^{(k)}(x, t, c) + \cdots,$$

satisfying (i) \sim (iv) below.

(i) $x_\ell^{(k)}(x, t, c)$ ($k \geq 0, \ell \geq 0$) are holomorphic in $x \in \mathcal{U}$ and also in t .

(ii) $x_0^{(0)}(x, t, c)$ satisfies that

$$x_0^{(0)}(\lambda_0, t, c) = 0, \quad (6.3)$$

$$\frac{\partial x_0^{(0)}}{\partial x}(\lambda_0, t, c) \neq 0, \quad (6.4)$$

(iii)

$$x_0^{(k)}(x, t, c) = 0 \quad (k \geq 1). \quad (6.5)$$

(iv)

$$Q_{\text{II}}(x, t, c, \eta; \alpha) = \left(\frac{\partial x_{\text{II}}}{\partial x} \right)^2 Q_{\text{can}}(x_{\text{II}}, E, \sigma, \rho, \eta) - \frac{1}{2} \eta^{-2} \{x_{\text{II}}; x\}, \quad (6.6)$$

where

$$\sigma = \sigma(t, c, \eta; \alpha) = \eta^{\frac{1}{2}} x_{\text{II}}(\lambda(t, c, \eta; \alpha), t, c, \eta; \alpha), \quad (6.7)$$

$$\begin{aligned} \rho = \rho(t, c, \eta; \alpha) &= -\eta^{\frac{1}{2}} \frac{\nu(t, c, \eta; \alpha)}{\frac{\partial x_{\text{II}}}{\partial x}(\lambda(t, c, \eta; \alpha), t, c, \eta; \alpha)} \\ &\quad - \frac{3}{4} \eta^{-\frac{1}{2}} \frac{\frac{\partial^2 x_{\text{II}}}{\partial x^2}(\lambda(t, c, \eta; \alpha), t, c, \eta; \alpha)}{\left(\frac{\partial x_{\text{II}}}{\partial x}(\lambda(t, c, \eta; \alpha), t, c, \eta; \alpha) \right)^2}, \end{aligned} \quad (6.8)$$

$$E = E(t, c, \eta; \alpha) = \rho(t, c, \eta; \alpha)^2 - 4\sigma(t, c, \eta; \alpha)^2, \quad (6.9)$$

and $\{x_{\text{II}}; x\}$ denotes the Schwarzian derivative, i.e.,

$$\{x_{\text{II}}; x\} = \frac{\partial^3 x_{\text{II}}}{\partial x^3} \Big/ \frac{\partial x_{\text{II}}}{\partial x} - \frac{3}{2} \left(\frac{\partial^2 x_{\text{II}}}{\partial x^2} \Big/ \frac{\partial x_{\text{II}}}{\partial x} \right)^2.$$

Remark 6.1. We note that [AKT1, Theorem3.1] and [KT3, Theorem2.1] deal with the case where a 2-parameter solution of (P_{II}) substituted into the coefficients of (SL_{II}) . Although 1-parameter solutions are not discussed in those papers, the proof of Theorem 6.1 can be done in a similar manner.

Remark 6.2. In what follows we abbreviate $x_0^{(0)}$ to x_0 for simplicity. The coefficients $x_\ell^{(k)}(x, t, c)$ are uniquely determined once the branch of

$$x_0(x, t, c) = \left[\int_{\lambda_0}^x \sqrt{Q_0(x, t, c)} dx \right]^{\frac{1}{2}} \quad (6.10)$$

is fixed. We adopt the branch in such a way that

$$x_0(x, t, c) \sim \frac{1}{\sqrt{2}} \Delta^{\frac{1}{4}}(x - \lambda_0) \quad (6.11)$$

holds as x tends to λ_0 , where the branch of $\Delta^{\frac{1}{4}}$ is taken to be the same as in the expressions (3.2) and (3.4) of $\lambda^{(1)}$.

We write down some properties of the formal series E , σ and ρ . These properties are used in the construction of t_{II} (in Proposition 6.3) and the computations of the Stokes multipliers of (SL_{II}) .

Proposition 6.1.

$$E(t, c, \eta; \alpha) = 0. \quad (6.12)$$

Proof. It is shown in [AKT1, (3.33)] that the following holds:

$$\frac{E}{4} = \text{Res}_{x=\lambda_0} S_{\text{odd}}. \quad (6.13)$$

Because $\text{Res}_{x=\lambda_0} S_{\text{odd}}$ is independent of t by (4.27), the coefficients of $e^{k\eta\phi_{\text{II}}}$ in

$$\text{Res}_{x=\lambda_0} S_{\text{odd}} = \text{Res}_{x=\lambda_0} S_{\text{odd}}^{(0)} + \alpha\eta^{-\frac{1}{2}} \text{Res}_{x=\lambda_0} S_{\text{odd}}^{(1)} e^{\eta\phi_{\text{II}}} + (\alpha\eta^{-\frac{1}{2}})^2 \text{Res}_{x=\lambda_0} S_{\text{odd}}^{(2)} e^{2\eta\phi_{\text{II}}} + \dots$$

must vanish for all $k \geq 1$. Furthermore, $\text{Res}_{x=\lambda_0} S_{\text{odd}}^{(0)} = 0$ follows from (ii) of Lemma 6.1 below. Thus we have (6.12). \square

The formal series σ and ρ of (6.7) and (6.8) are expanded as follows:

$$\sigma(t, c, \eta; \alpha) = \eta^{\frac{1}{2}} \{ \sigma^{(0)}(t, c, \eta) + \alpha\eta^{-\frac{1}{2}} \sigma^{(1)}(t, c, \eta) e^{\eta\phi_{\text{II}}} + (\alpha\eta^{-\frac{1}{2}})^2 \sigma^{(2)}(t, c, \eta) e^{2\eta\phi_{\text{II}}} + \dots \}, \quad (6.14)$$

$$\rho(t, c, \eta; \alpha) = \eta^{\frac{1}{2}} \{ \rho^{(0)}(t, c, \eta) + \alpha\eta^{-\frac{1}{2}} \rho^{(1)}(t, c, \eta) e^{\eta\phi_{\text{II}}} + (\alpha\eta^{-\frac{1}{2}})^2 \rho^{(2)}(t, c, \eta) e^{2\eta\phi_{\text{II}}} + \dots \}, \quad (6.15)$$

$$\begin{aligned} \sigma^{(k)}(t, c, \eta) &= \sigma_0^{(k)}(t, c) + \eta^{-1} \sigma_1^{(k)}(t, c) + \eta^{-2} \sigma_2^{(k)}(t, c) + \dots, \\ \rho^{(k)}(t, c, \eta) &= \rho_0^{(k)}(t, c) + \eta^{-1} \rho_1^{(k)}(t, c) + \eta^{-2} \rho_2^{(k)}(t, c) + \dots. \end{aligned}$$

We know that

$$\rho^{(0)}(t, c, \eta)^2 - 4\sigma^{(0)}(t, c, \eta)^2 = 0 \quad (6.16)$$

as a consequence of Proposition 6.1.

Proposition 6.2.

$$\sigma^{(0)}(t, c, \eta) = \rho^{(0)}(t, c, \eta) = 0. \quad (6.17)$$

To show Proposition 6.2, we recall the following lemma:

Lemma 6.1 ([KT4, Theorem 4.4], [KT2, Theorem 1.1, Theorem 1.2, Proposition 1.4]). (i) *There exists a formal power series $z(x, t, c, \eta) = z_0(x, t, c) + \eta^{-1} z_1(x, t, c) + \eta^{-2} z_2(x, t, c) + \dots$ whose coefficients are holomorphic at $x = \lambda_0$ satisfying*

$$Q_{\text{II}}^{(0)}(x, t, c, \eta) = \left(\frac{\partial z}{\partial x}(x, t, c, \eta) \right)^2 \left\{ 4z(x, t, c, \eta)^2 + \eta^{-2} \frac{3}{4z(x, t, c, \eta)^2} \right\} - \frac{1}{2} \eta^{-2} \{ z(x, t, c, \eta); x \}. \quad (6.18)$$

The coefficients $z_\ell(x, t, c)$ are uniquely determined once we fix the branch of

$$z_0(x, t, c) = \left[\int_{\lambda_0}^x \sqrt{Q_0(x, t, c)} dx \right]^{\frac{1}{2}}. \quad (6.19)$$

Moreover,

$$z_1(x, t, c) = 0. \quad (6.20)$$

(ii) *All the coefficients of the formal power series $S_{\text{odd}}^{(0)}$ and $S_{\text{odd}}^{(0)}/(x - \lambda^{(0)})$ are holomorphic at $x = \lambda_0$.*

(iii) *The formal series $z(x, t, c, \eta)$ in (i) satisfies the following:*

$$S_{\text{odd}}^{(0)}(x, t, c, \eta) = 2\eta z(x, t, c, \eta) \frac{\partial z}{\partial x}(x, t, c, \eta). \quad (6.21)$$

Since

$$S_{\text{odd}}^{(0)}(\lambda^{(0)}(t, c, \eta), t, c, \eta) = 0$$

follows from (ii) of Lemma 6.1, we obtain

$$z(\lambda^{(0)}(t, c, \eta), t, c, \eta) = 0 \quad (6.22)$$

from (6.21) and the invertibility of the formal power series $\frac{\partial z}{\partial x}(\lambda^{(0)})$. Now we show that the following lemma which is a generalization of Proposition 6.2:

Lemma 6.2. *If the branch of z_0 is taken as $z_0 = x_0$, then we have*

$$x_\ell^{(0)}(x, t, c) = z_\ell(x, t, c), \quad (6.23)$$

$$\sigma_\ell^{(0)}(t, c) = \rho_\ell^{(0)}(t, c) = 0, \quad (6.24)$$

for all $\ell \geq 0$.

Proof of Lemma 6.2. We prove Lemma 6.2 by induction. Because $\sigma_0^{(0)} = x_0(\lambda_0) = 0$ and $\rho_0^{(0)^2} - 4\sigma_0^{(0)^2} = 0$ follow from (6.10) and (6.16), the claim for $\ell = 0$ holds. Next we assume that the claims are true for $\ell = 0, \dots, L-1$ for a positive integer $L \geq 1$. Since the formal power series $x_\Pi^{(0)}(x, t, c, \eta)$ and $z(x, t, c, \eta)$ satisfy

$$\begin{aligned} Q_\Pi^{(0)}(x, t, c, \eta) &= \left(\frac{\partial x_\Pi^{(0)}}{\partial x}(x, t, c, \eta) \right)^2 Q_{\text{can}}(x_\Pi^{(0)}(x, t, c, \eta), E^{(0)}(t, c, \eta), \sigma^{(0)}(t, c, \eta), \rho^{(0)}(t, c, \eta), \eta) \\ &\quad - \frac{1}{2} \eta^{-2} \{x_\Pi^{(0)}(x, t, c, \eta); x\}, \end{aligned} \quad (6.25)$$

and (6.18) respectively, $x_L^{(0)}$ and z_L satisfy the following differential equations respectively:

$$8x_0 \frac{\partial x_0}{\partial x} \left(x_0 \frac{\partial x_L^{(0)}}{\partial x} + x_L^{(0)} \frac{\partial x_0}{\partial x} \right) = r_L(x, t, c), \quad (6.26)$$

$$8z_0 \frac{\partial z_0}{\partial x} \left(z_0 \frac{\partial z_L}{\partial x} + z_L \frac{\partial z_0}{\partial x} \right) = \hat{r}_L(x, t, c). \quad (6.27)$$

Here r_L (resp. \hat{r}_L) is written by x_0, \dots, x_{L-1} , $\sigma_0^{(0)}, \dots, \sigma_{L-1}^{(0)}$ and $\rho_0^{(0)}, \dots, \rho_{L-1}^{(0)}$ (resp. z_0, \dots, z_{L-1} , $\sigma_0^{(0)}, \dots, \sigma_{L-1}^{(0)}$ and $\rho_0^{(0)}, \dots, \rho_{L-1}^{(0)}$). Therefore $r_L = \hat{r}_L$ holds under the assumption of the induction. Hence it follows from (6.26) and (6.27) that $x_0(x_L^{(0)} - z_L)$ equals some constant which is independent of x . The holomorphy of $x_L^{(0)}$ and z_L at $x = \lambda_0$ implies that the constant must be 0. Thus we obtain $x_L^{(0)} = z_L$, and $\sigma_L^{(0)} = \rho_L^{(0)} = 0$ follows from (6.22) and (6.16). \square

We note that

$$x_1^{(0)}(x, t, c) = 0, \quad (6.28)$$

$$S_{\text{odd}}^{(0)}(x, t, c, \eta) = 2\eta x_\Pi^{(0)}(x, t, c, \eta) \frac{\partial x_\Pi^{(0)}}{\partial x}(x, t, c, \eta), \quad (6.29)$$

follow from Lemma 6.1 and Lemma 6.2.

Unfortunately, Theorem 6.1 is not sufficient to determine the connection formula for $\psi_{\pm, \text{IM}}$ on Stokes curves emanating from $x = \lambda_0$ in all orders. For that purpose we consider an extended transformation of the simultaneous equations (SL_Π) and (D_Π) , which is constructed in [KT3]. (See Proposition 6.4 below.) We introduce the formal series t_Π which plays a role of the transformation of the independent variable of (H_{can}) .

Proposition 6.3 ([KT3, Lemma 3.3]). *There exists a formal series*

$$\begin{aligned} t_{\text{II}} &= t_{\text{II}}(t, c, \eta; \alpha) \\ &= t_{\text{II}}^{(0)}(t, c, \eta) + \alpha \eta^{-\frac{1}{2}} t_{\text{II}}^{(1)}(t, c, \eta) e^{\eta \phi_{\text{II}}} + (\alpha \eta^{-\frac{1}{2}})^2 t_{\text{II}}^{(2)}(t, c, \eta) e^{2\eta \phi_{\text{II}}} + \dots, \end{aligned} \quad (6.30)$$

$$t_{\text{II}}^{(k)}(t, c, \eta) = t_0^{(k)}(t, c) + \eta^{-1} t_1^{(k)}(t, c) + \eta^{-2} t_2^{(k)}(t, c) + \dots, \quad (6.31)$$

satisfying (i) and (ii) below.

(i)

$$t_0^{(0)}(t, c) = \frac{1}{2} \phi_{\text{II}}(t, c). \quad (6.32)$$

$$t_1^{(0)}(t, c) = 0. \quad (6.33)$$

$$t_0^{(k)}(t, c) = 0 \quad (k \geq 1). \quad (6.34)$$

(ii)

$$\sigma(t, c, \eta; \alpha) = \frac{1}{\sqrt{2}} \alpha e^{2\eta t_{\text{II}}(t, c, \eta; \alpha)}. \quad (6.35)$$

$$\rho(t, c, \eta; \alpha) = -\sqrt{2} \alpha e^{2\eta t_{\text{II}}(t, c, \eta; \alpha)}. \quad (6.36)$$

Remark 6.3. The right-hand sides of (6.35) and (6.36) are the formal series obtained from the solution (6.1) of (H_{can}) through the substitution $\tilde{t} \mapsto t_{\text{II}}$ and $(A, B) \mapsto (\frac{1}{\sqrt{2}}\alpha, 0)$. The choice of the parameters originate from Proposition 6.2 and the following relations:

$$\begin{aligned} \sigma_0^{(1)}(t, c) &= x_0^{(1)}(\lambda_0, t, c) + \frac{\partial x_0^{(0)}}{\partial x}(\lambda_0, t, c) \lambda_0^{(1)}(t, c) = \frac{1}{\sqrt{2}}, \\ \rho_0^{(1)}(t, c) &= \frac{\nu_0(t, c) \left(\frac{\partial x_0^{(1)}}{\partial x}(\lambda_0, t, c) + \frac{\partial^2 x_0}{\partial x^2}(\lambda_0, t) \lambda_0^{(1)}(t, c) \right)}{\left(\frac{\partial x_0}{\partial x}(\lambda_0, t, c) \right)^2} - \frac{\nu_0^{(1)}(t, c)}{\frac{\partial x_0}{\partial x}(\lambda_0, t, c)} = -\sqrt{2}. \end{aligned}$$

In what follows we abbreviate $t_0^{(0)}(t, c)$ to $t_0(t, c)$. The formal power series $t_{\text{II}}^{(0)}$ is determined by the following relation:

$$\sigma^{(1)}(t, c, \eta) = \frac{1}{\sqrt{2}} \exp \left(2\eta (t_{\text{II}}^{(0)}(t, c, \eta) - t_0(t, c)) \right). \quad (6.37)$$

(We note that $\sigma^{(0)}(t, c, \eta) = 0$ by Proposition 6.2.) The relation

$$1 + \alpha \eta^{-\frac{1}{2}} \frac{\sigma^{(2)}(t, c, \eta)}{\sigma^{(1)}(t, c, \eta)} e^{\eta \phi_{\text{II}}} + (\alpha \eta^{-\frac{1}{2}})^2 \frac{\sigma^{(3)}(t, c, \eta)}{\sigma^{(1)}(t, c, \eta)} e^{2\eta \phi_{\text{II}}} + \dots = \exp \left(2\eta (t_{\text{II}}(t, c, \eta) - t_{\text{II}}^{(0)}(t, c, \eta)) \right) \quad (6.38)$$

also determines $t_{\text{II}}^{(k)}$ ($k \geq 1$) uniquely.

Making use of the above transformation theory, we obtain a correspondence between WKB solutions of (SL_{II}) and (Can) satisfying their deformation equations.

Proposition 6.4 ([KT3, Proposition 3.1]). *Let $\tilde{\psi}(\tilde{x}, \tilde{t}, \eta; \alpha)$ be a WKB solution of (Can) and (D_{can}) with the solution*

$$\begin{cases} \tilde{\sigma}(\tilde{t}, \eta; \alpha) = \frac{1}{\sqrt{2}} \alpha e^{2\eta \tilde{t}}, \\ \tilde{\rho}(\tilde{t}, \eta; \alpha) = -\sqrt{2} \alpha e^{2\eta \tilde{t}}, \end{cases} \quad (6.39)$$

of (H_{can}) substituted into their coefficients. If we define

$$\psi(x, t, c, \eta; \alpha) = \left(\frac{\partial x_{\text{II}}}{\partial x}(x, t, c, \eta; \alpha) \right)^{-\frac{1}{2}} \tilde{\psi}(x_{\text{II}}(x, t, c, \eta; \alpha), t_{\text{II}}(t, c, \eta; \alpha), \eta; \alpha), \quad (6.40)$$

then ψ satisfies both (SL_{II}) and (D_{II}) in a neighborhood of $x = \lambda_0$.

Proposition 6.4 can be verified easily by using the relation

$$\frac{1}{2(x-\lambda)} \frac{\partial x_{\text{II}}}{\partial x} - \frac{\partial x_{\text{II}}}{\partial t} - \frac{1}{2(x_{\text{II}} - \eta^{-\frac{1}{2}}\sigma)} \frac{\partial t_{\text{II}}}{\partial t} = 0, \quad (6.41)$$

which is proved in [KT3, (3.52)].

6.2 WKB solutions of (Can) and (D_{can})

In order to use Proposition 6.4, we construct WKB solutions which satisfy both (Can) and (D_{can}) with (6.39) are substituted into the coefficients. In what follows we designate

$$\begin{aligned} Q_{\text{can}}(\tilde{x}, \tilde{t}, \eta; \alpha) &= Q_{\text{can}}(\tilde{x}, \tilde{E}(\tilde{t}, \eta; \alpha), \tilde{\sigma}(\tilde{t}, \eta; \alpha), \tilde{\rho}(\tilde{t}, \eta; \alpha), \eta) \\ &= 4\tilde{x}^2 - \eta^{-1} \frac{\sqrt{2}\alpha\eta^{-\frac{1}{2}}e^{2\eta\tilde{t}}}{\tilde{x} - \frac{1}{\sqrt{2}}\alpha\eta^{-\frac{1}{2}}e^{2\eta\tilde{t}}} + \eta^{-2} \frac{3}{4\left(\tilde{x} - \frac{1}{\sqrt{2}}\alpha\eta^{-\frac{1}{2}}e^{2\eta\tilde{t}}\right)^2}, \\ A_{\text{can}}(\tilde{x}, \tilde{t}, \eta; \alpha) &= \frac{1}{2\left(\tilde{x} - \frac{1}{\sqrt{2}}\alpha\eta^{-\frac{1}{2}}e^{2\eta\tilde{t}}\right)}. \end{aligned}$$

(Note that $\tilde{E}(\tilde{t}, \eta; \alpha) = \tilde{\rho}(\tilde{t}, \eta; \alpha)^2 - 4\tilde{\sigma}(\tilde{t}, \eta; \alpha)^2 = 0$.) Q_{can} is expanded as follows:

$$Q_{\text{can}}(\tilde{x}, \tilde{t}, \eta; \alpha) = Q_{\text{can}}^{(0)}(\tilde{x}, \eta) + \alpha\eta^{-\frac{1}{2}}Q_{\text{can}}^{(1)}(\tilde{x}, \eta)e^{2\eta\tilde{t}} + (\alpha\eta^{-\frac{1}{2}})^2Q_{\text{can}}^{(2)}(\tilde{x}, \eta)e^{4\eta\tilde{t}} + \dots, \quad (6.42)$$

$$Q_{\text{can}}^{(0)}(\tilde{x}, \eta) = 4\tilde{x}^2 + \eta^{-2} \frac{3}{4\tilde{x}^2}, \quad (6.43)$$

$$Q_{\text{can}}^{(k)}(\tilde{x}, \eta) = -\eta^{-1} \left(\frac{1}{\sqrt{2}}\right)^{k-2} \frac{1}{\tilde{x}^k} + \eta^{-2} \left(\frac{1}{\sqrt{2}}\right)^k \frac{3(k+1)}{4\tilde{x}^{k+2}} \quad (k \geq 1). \quad (6.44)$$

As in Section 4, we can construct WKB solutions of (Can) in the following form:

$$\tilde{\psi}_{\pm}(\tilde{x}, \tilde{t}, \eta; \alpha) = \frac{1}{\sqrt{\tilde{S}_{\text{odd}}}} \exp \pm \left\{ \eta \int_0^{\tilde{x}} \tilde{S}_{-1} d\tilde{x} + \int_{\infty}^{\tilde{x}} (\tilde{S}_{\text{odd}} - \eta\tilde{S}_{-1}) d\tilde{x} \right\}, \quad (6.45)$$

where

$$\tilde{S}_{\text{odd}} = \tilde{S}_{\text{odd}}(\tilde{x}, \tilde{t}, \eta; \alpha) = \tilde{S}_{\text{odd}}^{(0)}(\tilde{x}, \eta) + \alpha\eta^{-\frac{1}{2}}\tilde{S}_{\text{odd}}^{(1)}(\tilde{x}, \eta)e^{2\eta\tilde{t}} + (\alpha\eta^{-\frac{1}{2}})^2\tilde{S}_{\text{odd}}^{(2)}(\tilde{x}, \eta)e^{4\eta\tilde{t}} + \dots \quad (6.46)$$

is the odd part (in the sense of Remark 2.1) of a formal solution

$$\tilde{S} = \tilde{S}(\tilde{x}, \tilde{t}, \eta; \alpha) = \tilde{S}^{(0)}(\tilde{x}, \eta) + \alpha\eta^{-\frac{1}{2}}\tilde{S}^{(1)}(\tilde{x}, \eta)e^{2\eta\tilde{t}} + (\alpha\eta^{-\frac{1}{2}})^2\tilde{S}^{(2)}(\tilde{x}, \eta)e^{4\eta\tilde{t}} + \dots \quad (6.47)$$

of the associated Riccati equation of (Can)

$$\tilde{S}^2 + \frac{\partial \tilde{S}}{\partial \tilde{x}} = \eta^2 Q_{\text{can}}(\tilde{x}, \tilde{t}, \eta; \alpha),$$

$\tilde{S}_{\text{odd}}^{(k)}$ and $\tilde{S}^{(k)}$ are formal power series of η^{-1} of the form

$$\tilde{S}_{\text{odd}}^{(k)}(\tilde{x}, \eta) = \eta\tilde{S}_{\text{odd},-1}^{(k)}(\tilde{x}) + \tilde{S}_{\text{odd},0}^{(k)}(\tilde{x}) + \eta^{-1}\tilde{S}_{\text{odd},1}^{(k)}(\tilde{x}) + \dots \quad (k \geq 0),$$

$$\tilde{S}^{(k)}(\tilde{x}, \eta) = \eta\tilde{S}_{-1}^{(k)}(\tilde{x}) + \tilde{S}_0^{(k)}(\tilde{x}) + \eta^{-1}\tilde{S}_1^{(k)}(\tilde{x}) + \dots \quad (k \geq 0),$$

and

$$\tilde{S}_{-1}(\tilde{x}) = \tilde{S}_{-1}^{(0)}(\tilde{x}) = 2\tilde{x}. \quad (6.48)$$

It is easy to check the integral in (6.45) is well-defined. Since $\tilde{S}^{(k)}$ ($k \geq 0$) satisfy the differential equations

$$\tilde{S}^{(0)2} + \frac{\partial \tilde{S}^{(0)}}{\partial \tilde{x}} = \eta^2 Q_{\text{can}}^{(0)}(\tilde{x}, \eta), \quad (6.49)$$

$$2\tilde{S}^{(0)}\tilde{S}^{(k)} + \sum_{k_1+k_2=k, k_j < k} \tilde{S}^{(k_1)}\tilde{S}^{(k_2)} + \frac{\partial \tilde{S}^{(k)}}{\partial \tilde{x}} = \eta^2 Q_{\text{can}}^{(k)}(\tilde{x}, \eta) \quad (k \geq 1), \quad (6.50)$$

we obtain

$$\tilde{S}^{(0)}(\tilde{x}, \eta) = 2\eta\tilde{x} - \frac{1}{2\tilde{x}}, \quad (6.51)$$

$$\tilde{S}_{\text{odd}}^{(0)}(\tilde{x}, \eta) = 2\eta\tilde{x}, \quad (6.52)$$

$$\tilde{S}_{-1}^{(k)}(\tilde{x}) = 0 \quad (k \geq 1), \quad (6.53)$$

from (6.43) and (6.44).

Lemma 6.3 ([T1, Lemma2]). *The formal series $e^{\pm\eta\tilde{t}}\tilde{\psi}_{\pm}$ satisfy both (Can) and (D_{can}).*

Remark 6.4. In view of (6.46), (6.48) and (6.53), $e^{\pm\eta\tilde{t}}\tilde{\psi}_{\pm}$ are expanded as follows:

$$e^{\pm\eta\tilde{t}}\tilde{\psi}_{\pm}(\tilde{x}, \tilde{t}, \eta) = \tilde{\psi}_{\pm}^{(0)}(\tilde{x}, \tilde{t}, \eta) + \alpha\eta^{-\frac{1}{2}}\tilde{\psi}_{\pm}^{(1)}(\tilde{x}, \tilde{t}, \eta)e^{2\eta\tilde{t}} + (\alpha\eta^{-\frac{1}{2}})^2\tilde{\psi}_{\pm}^{(2)}(\tilde{x}, \tilde{t}, \eta)e^{4\eta\tilde{t}} + \dots, \quad (6.54)$$

$$\tilde{\psi}_{\pm}^{(k)}(\tilde{x}, \tilde{t}, \eta) = \eta^{-\frac{1}{2}} \left\{ \tilde{\psi}_{\pm,0}^{(k)}(\tilde{x}) + \eta^{-1}\tilde{\psi}_{\pm,1}^{(k)}(\tilde{x}) + \eta^{-2}\tilde{\psi}_{\pm,2}^{(k)}(\tilde{x}) + \dots \right\} \exp \pm \eta(\tilde{t} + \tilde{x}^2). \quad (6.55)$$

Especially,

$$\tilde{\psi}_{\pm}^{(0)}(\tilde{x}, \tilde{t}, \eta) = \frac{1}{\sqrt{\tilde{S}_{\text{odd}}^{(0)}}} \exp \pm \left(\int_{\infty}^{\tilde{x}} (\tilde{S}_{\text{odd}}^{(0)} - \eta\tilde{S}_{-1}) d\tilde{x} \right) \exp \pm \eta(\tilde{t} + \tilde{x}^2). \quad (6.56)$$

Making use of Proposition 6.4 and Lemma 6.3, we know that

$$e^{\pm\eta t_{\text{II}}(t, c, \eta; \alpha)} \left(\frac{\partial x_{\text{II}}}{\partial x}(x, t, c, \eta; \alpha) \right)^{-\frac{1}{2}} \tilde{\psi}_{\pm}(x_{\text{II}}(x, t, c, \eta; \alpha), t_{\text{II}}(t, c, \eta; \alpha), \eta; \alpha)$$

satisfy both (SL_{II}) and (D_{II}) near $x = \lambda_0$. Therefore, there exist a formal power series $C_{\pm} = C_{\pm}(c, \eta)$ of η^{-1} whose coefficients are independent of x and t such that

$$\psi_{\pm, \text{IM}} = C_{\pm} e^{\pm\eta t_{\text{II}}} \left(\frac{\partial x_{\text{II}}}{\partial x} \right)^{-\frac{1}{2}} \tilde{\psi}_{\pm}(x_{\text{II}}, t_{\text{II}}, \eta; \alpha) \quad (6.57)$$

hold.

Lemma 6.4.

$$C_{\pm} = \exp \pm \left\{ \frac{1}{2}U^{(0)} + \int_{\infty}^x (S_{\text{odd}}^{(0)} - \eta S_{-1}) dx - \eta(t_{\text{II}}^{(0)} - t_0) - \eta(x_{\text{II}}^{(0)2} - x_0^2) \right\}. \quad (6.58)$$

Remark 6.5. Precisely speaking, we need to specify the normalization of $\psi_{\pm, \text{IM}}$ (i.e., the choices of the path of integration of $\int_{a_1}^x S_{-1} dx$ and $\int_{\infty}^x (S_{\text{odd}} - \eta S_{-1}) dx$ in (4.11)). (6.58) is the result when the normalization of $\psi_{\pm, \text{IM}}$ is taken as in Figure 6.1, where the red and blue curves designate the path of integration of $\int_{a_1}^x S_{-1} dx$ and $\int_{\infty}^x (S_{\text{odd}} - \eta S_{-1}) dx$ respectively.

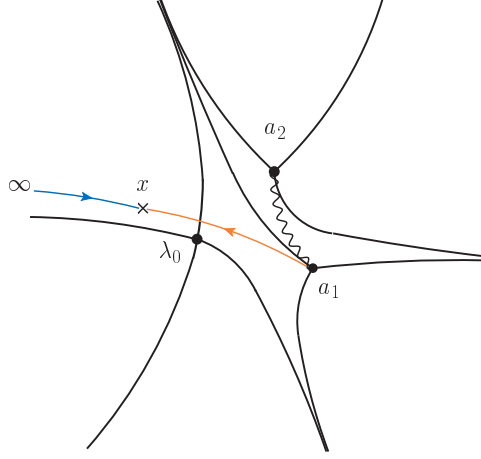


Figure 6.1: Normalization of $\psi_{\pm, \text{IM}}$.

Proof of Lemma 6.4. Expanding the both sides of (6.57) to the form like (4.38) and comparing the parts which do not contain $e^{k\eta\phi_{\text{II}}}$ ($k \geq 1$), we obtain

$$\psi_{\pm}^{(0)} = C_{\pm} \left(\frac{\partial x_{\text{II}}^{(0)}}{\partial x} \right)^{-\frac{1}{2}} \tilde{\psi}_{\pm}(x_{\text{II}}^{(0)}, t_{\text{II}}^{(0)}, \eta). \quad (6.59)$$

Note that, since C_{\pm} is independent of t , it does not contain $e^{k\eta\phi_{\text{II}}}$ ($k \geq 1$), either. By (4.9), we have

$$\begin{aligned} t_0(t, c) + x_0(x, t, c)^2 &= \frac{1}{2} \int_{\tau_1}^t \sqrt{\Delta(t, c)} dt + \int_{\lambda_0}^x \sqrt{Q_0(x, t, c)} dx \\ &= \frac{1}{2} \int_{\tau_1}^t \sqrt{\Delta(t, c)} dt + \int_{\lambda_0}^{a_1} \sqrt{Q_0(x, t, c)} dx + \int_{a_1}^x \sqrt{Q_0(x, t, c)} dx \\ &= \int_{a_1}^x \sqrt{Q_0(x, t, c)} dx = \int_{a_1}^x S_{-1}(x, t, c) dx. \end{aligned} \quad (6.60)$$

(As in the proof of Proposition 4.1, we note that the sign of the right-hand side of (4.9) is +.) Making use of (4.39), (6.56), (6.29), (6.52) and (6.60), (6.58) is derived from (6.59) directly. \square

Let $Z = Z(c, \eta)$ be a formal power series defined by

$$\frac{1}{2} U^{(0)} + \int_{\infty}^x (S_{\text{odd}}^{(0)} - \eta S_{-1}) dx - \eta(t_{\text{II}}^{(0)} - t_0) - \eta(x_{\text{II}}^{(0)^2} - x_0^2).$$

In the case where a 1-parameter solution of (H_{II}) is normalized at $t = \infty$, we can determine Z explicitly with the aid of the results presented in Appendix.

Proposition 6.5. *Assume that the 1-parameter solution λ_{∞} normalized at $t = \infty$ is substituted into the coefficients of (SL_{II}) and (D_{II}) . Then*

$$Z = 0. \quad (6.61)$$

Proof. Z has the form

$$Z(c, \eta) = Z_0 + Z_1(c\eta)^{-1} + Z_2(c\eta)^{-2} + \dots$$

for some $Z_{\ell} \in \mathbb{C}$ which is independent of c because of (A.18), (A.22) and Proposition A.1 in Appendix A. If $\lambda_{\infty}(t, c, \eta; \alpha)$ is substituted into the coefficients of (SL_{II}) and (D_{II}) ,

then the coefficients of $\eta^{-\ell}$ in Z are holomorphic in c for all $\ell \geq 0$ by Proposition B.1 in Appendix B. Thus we have $Z_\ell = 0$ ($\ell \geq 1$). Furthermore, we can confirm that $Z_0 = 0$ easily. \square

Therefore we have the following correspondence between $\psi_{\pm, \text{IM}}$ and $\tilde{\psi}_{\pm}$ when the 1-parameter solution λ_∞ normalized at $t = \infty$ is substituted:

$$\psi_{\pm, \text{IM}}(x, t, c, \eta; \alpha) = e^{\pm \eta t_{\text{II}}(t, c, \eta; \alpha)} \left(\frac{\partial x_{\text{II}}}{\partial x}(x, t, c, \eta; \alpha) \right)^{-\frac{1}{2}} \tilde{\psi}_{\pm}(x_{\text{II}}(x, t, c, \eta; \alpha), t_{\text{II}}(t, c, \eta; \alpha), \eta; \alpha). \quad (6.62)$$

We then expect that the connection formulas for $\psi_{\pm, \text{IM}}$ on Stokes curves emanating from the double turning point $x = \lambda_0$ should be derived from this relation (6.62) and the connection formulas for $\tilde{\psi}_{\pm}$. The latter is explicitly described in the following Proposition 6.6 which is proved in [T1, §3].

Proposition 6.6 ([T1, Proposition 4]). *Let $\tilde{\psi}_{\pm}^{\text{J}}$ be the Borel sum of $\tilde{\psi}_{\pm}$ in the region J in Figure 6.2 ($\text{J} = 0, \text{I}, \text{II}, \text{III}$). Then on each Stokes curve the following connection formula holds:*

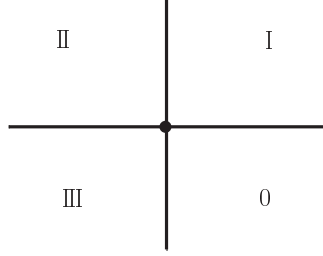


Figure 6.2: Stokes curves of (Can) .

$$\begin{cases} \tilde{\psi}_+^0 = \tilde{\psi}_+^{\text{I}} + \tilde{m}_{01} \tilde{\psi}_-^{\text{I}} \\ \tilde{\psi}_-^0 = \tilde{\psi}_-^{\text{I}} \end{cases} \quad (6.63)$$

$$\begin{cases} \tilde{\psi}_+^{\text{I}} = \tilde{\psi}_+^{\text{II}} \\ \tilde{\psi}_-^{\text{I}} = \tilde{\psi}_-^{\text{II}} + \tilde{m}_{12} \tilde{\psi}_+^{\text{II}} \end{cases} \quad (6.64)$$

$$\begin{cases} \tilde{\psi}_+^{\text{II}} = \tilde{\psi}_+^{\text{III}} + \tilde{m}_{23} \tilde{\psi}_-^{\text{III}} \\ \tilde{\psi}_-^{\text{II}} = \tilde{\psi}_-^{\text{III}} \end{cases} \quad (6.65)$$

$$\begin{cases} \tilde{\psi}_+^{\text{III}} = \tilde{\psi}_+^0 \\ \tilde{\psi}_-^{\text{III}} = \tilde{\psi}_-^0 + \tilde{m}_{30} \tilde{\psi}_+^0 \end{cases} \quad (6.66)$$

where

$$\tilde{m}_{01} = -i(\tilde{\rho} + 2\tilde{\sigma})\sqrt{\frac{\pi}{2}}, \quad (6.67)$$

$$\tilde{m}_{12} = (\tilde{\rho} - 2\tilde{\sigma})\sqrt{\frac{\pi}{2}}, \quad (6.68)$$

$$\tilde{m}_{23} = i(\tilde{\rho} + 2\tilde{\sigma})\sqrt{\frac{\pi}{2}}, \quad (6.69)$$

$$\tilde{m}_{30} = -(\tilde{\rho} - 2\tilde{\sigma})\sqrt{\frac{\pi}{2}}. \quad (6.70)$$

Remark 6.6. In Proposition 6.6 $\tilde{\sigma}$ and $\tilde{\rho}$ in the coefficients of (Can) should be understood as ordinary parameters (possibly depending on t and η), although $\sigma = \sigma(t, c, \eta; \alpha)$ and $\rho = \rho(t, c, \eta; \alpha)$ are infinite series in our situation (cf. Remark ?? below).

6.3 Computation of the Stokes multipliers of (SL_{II})

In this subsection we compute the Stokes multipliers of (SL_{II}) around $x = \infty$ before and after the degeneration of the Stokes geometry observed when $\arg c = \frac{\pi}{2}$. The symbols \oplus

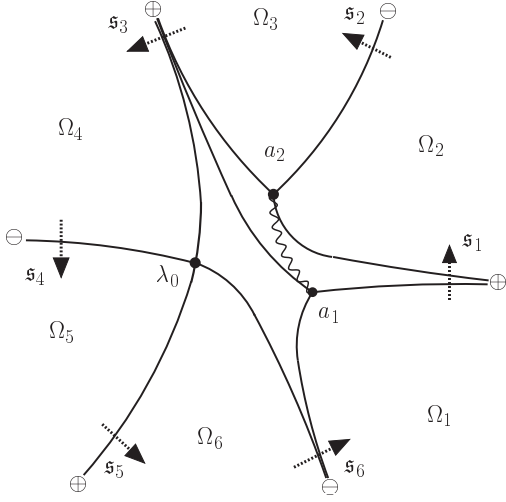


Figure 6.3: $\arg c = \frac{\pi}{2} - \varepsilon$

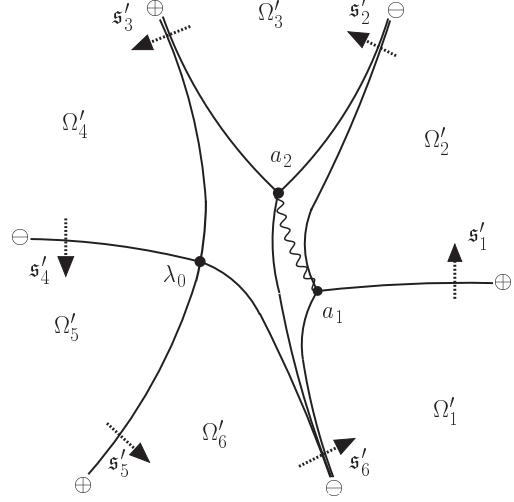


Figure 6.4: $\arg c = \frac{\pi}{2} + \varepsilon$

and \ominus in Figures 6.3 and 6.4 designate the “sign of Stokes curves” which is defined similarly to that for P -Stokes curves (cf. Remark 2.2). Note that it follows from Proposition 4.1 that $\operatorname{Re} \int_{a_1}^{a_2} \sqrt{Q_0} dx < 0$ holds near $\arg c = \frac{\pi}{2}$. Since a neighborhood of $x = \infty$ is divided into the six regions Ω_j and Ω'_j ($1 \leq j \leq 6$) by Stokes curves as in Figures 6.3 and 6.4, we obtain six Stokes multipliers around $x = \infty$ for $\arg c = \frac{\pi}{2} - \varepsilon$ and $\arg c = \frac{\pi}{2} + \varepsilon$, respectively. Let $\mathfrak{s}_j = \mathfrak{s}_j(c; \alpha)$ (resp. $\mathfrak{s}'_j = \mathfrak{s}'_j(c; \alpha)$) be the Stokes multipliers corresponding to the analytic continuation from Ω_j to Ω_{j+1} (resp. from Ω'_j to Ω'_{j+1}) ($1 \leq j \leq 6$). The results of the computations of the Stokes multipliers of (SL_{II}) by using the WKB solutions $(\psi_{+,IM}, \psi_{-,IM})$ are as follows:

Stokes multipliers of (SL_{II}) around $x = \infty$.

(i) If the 1-parameter solution substituted into the coefficients of (SL_{II}) and (D_{II}) is normalized at ∞ , then we obtain the following:

$$\begin{cases} \mathfrak{s}_1 = i (1 + e^{2\pi i c \eta}) e^{U-2V} \\ \mathfrak{s}_2 = i e^{-2\pi i c \eta} e^{2V-U} \\ \mathfrak{s}_3 = i (1 + e^{2\pi i c \eta}) e^{-2\pi i c \eta} e^{U-2V} \\ \mathfrak{s}_4 = -2\sqrt{\pi}\alpha \\ \mathfrak{s}_5 = 0 \\ \mathfrak{s}_6 = 2\sqrt{\pi}\alpha + i e^{2V-U}. \end{cases} \quad \begin{cases} \mathfrak{s}'_1 = i e^{U-2V} \\ \mathfrak{s}'_2 = i (1 + e^{2\pi i c \eta}) e^{-2\pi i c \eta} e^{2V-U} \\ \mathfrak{s}'_3 = i e^{-2\pi i c \eta} e^{U-2V} \\ \mathfrak{s}'_4 = -2\sqrt{\pi}\alpha \\ \mathfrak{s}'_5 = 0 \\ \mathfrak{s}'_6 = 2\sqrt{\pi}\alpha + i (1 + e^{2\pi i c \eta}) e^{2V-U}. \end{cases} \quad (6.71)$$

(ii) If the 1-parameter solution substituted into the coefficients of (SL_{II}) and (D_{II}) is

normalized at τ_1 , then we obtain the following:

$$\begin{cases} \mathfrak{s}_1 = i(1 + e^{2\pi i c \eta})e^{U-2V} \\ \mathfrak{s}_2 = i e^{-2\pi i c \eta} e^{2V-U} \\ \mathfrak{s}_3 = i(1 + e^{2\pi i c \eta})e^{-2\pi i c \eta} e^{U-2V} \\ \mathfrak{s}_4 = -2\sqrt{\pi} \alpha e^W \\ \mathfrak{s}_5 = 0 \\ \mathfrak{s}_6 = 2\sqrt{\pi} \alpha e^W + i e^{2V-U}. \end{cases} \quad \begin{cases} \mathfrak{s}'_1 = i e^{U-2V} \\ \mathfrak{s}'_2 = i(1 + e^{2\pi i c \eta})e^{-2\pi i c \eta} e^{2V-U} \\ \mathfrak{s}'_3 = i e^{-2\pi i c \eta} e^{U-2V} \\ \mathfrak{s}'_4 = -2\sqrt{\pi} \alpha e^W \\ \mathfrak{s}'_5 = 0 \\ \mathfrak{s}'_6 = 2\sqrt{\pi} \alpha e^W + i(1 + e^{2\pi i c \eta})e^{2V-U}. \end{cases} \quad (6.72)$$

Here α is the free parameter contained in the 1-parameter solution substituted into the coefficients of (SL_{II}) and (D_{II}) , $U = U(t, c, \eta; \alpha)$ is given by (4.32), $V = V(t, c, \eta; \alpha)$ is the Voros coefficient (5.1) of (SL_{II}) and $W = W(c, \eta)$ is the P -Voros coefficient (3.6).

To be precise, the Stokes multipliers are the Borel sum of \mathfrak{s}_j and \mathfrak{s}'_j in (6.71) and (6.72). From now on we demonstrate the computation of the Stokes multipliers of (SL_{II}) when $\arg c = \frac{\pi}{2} - \varepsilon$ and the 1-parameter solution substituted into the coefficients is normalized at ∞ . We demonstrate only the computations of \mathfrak{s}_1 and \mathfrak{s}_4 , since the other Stokes multipliers can be computed in similar ways. We note that, since the two normalizations of 1-parameter solutions introduced in Section 3 are related as (3.7), the result (6.72) of the computation when the substituted 1-parameter solution is normalized at $t = \tau_1$ is obtained from (6.71) by replacement $\alpha \mapsto \alpha e^W$.

Remark 6.7. In the computations of the Stokes multipliers the path of normalization of WKB solutions $\psi_{\pm, \text{IM}}$ are taken as in Figures 6.5 ~ 6.10. The red and blue curves in Figures 6.5 ~ 6.10 designate the paths of integration of $\int_{a_1}^x S_{-1} dx$ and $\int_{\infty}^x (S_{\text{odd}} - \eta S_{-1}) dx$ respectively.

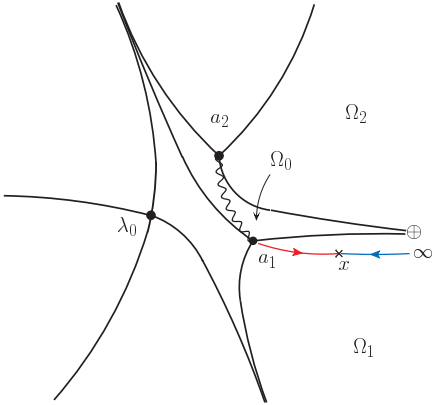


Figure 6.5: Normalization path for \mathfrak{s}_1 .

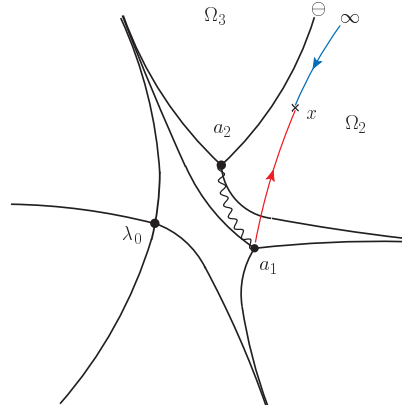


Figure 6.6: Normalization path for \mathfrak{s}_2 .

We denote the Borel sum of a WKB solution ψ in a region Ω by ψ^Ω .

- **Computation of \mathfrak{s}_1 .** Let Ω_0 be the region pinched by Ω_1 and Ω_2 , and ψ_{\pm, a_j} be a WKB solution of (SL_{II}) normalized at $x = a_j$:

$$\psi_{\pm, a_j} = \frac{1}{\sqrt{S_{\text{odd}}}} \exp\left(\pm \int_{a_j}^x S_{\text{odd}} dx\right) \quad (j = 1, 2).$$

Voros' connection formula ([KT4, §2, Theorem 2.23], [V]) says that the following relations

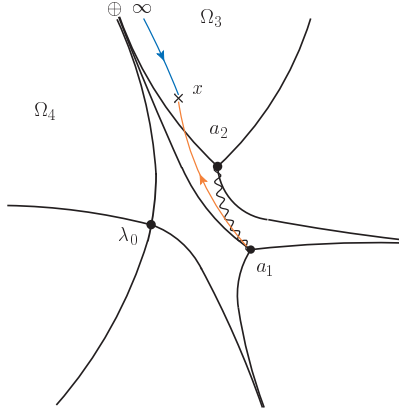


Figure 6.7: Normalization path for \mathfrak{s}_3 .

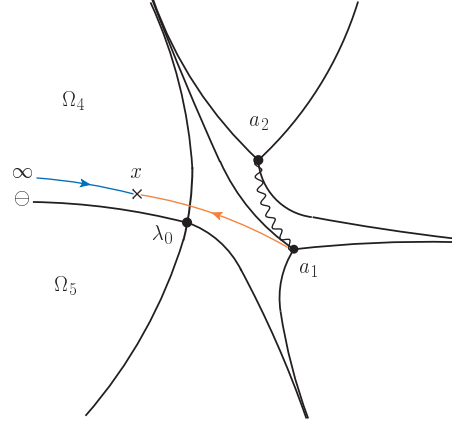


Figure 6.8: Normalization path for \mathfrak{s}_4 .

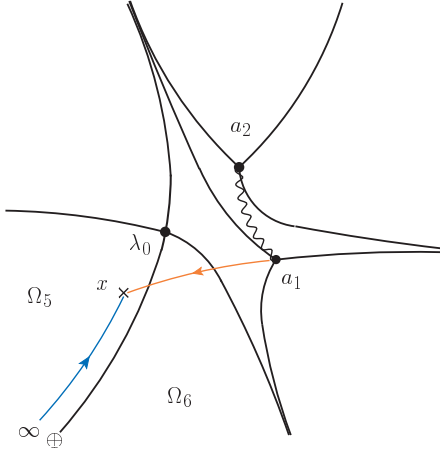


Figure 6.9: Normalization path for \mathfrak{s}_5 .

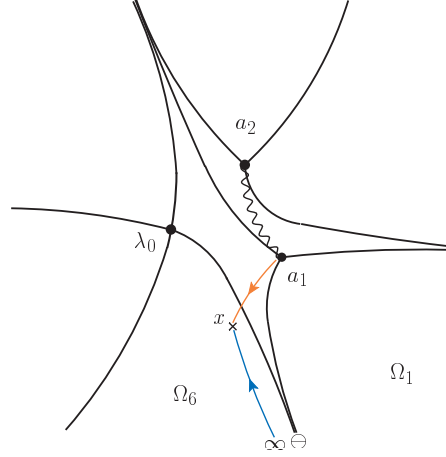


Figure 6.10: Normalization path for \mathfrak{s}_6 .

for the Borel sums of ψ_{\pm, a_j} hold on the Stokes curve emanating from $x = a_j$ ($j = 1, 2$):

$$\begin{cases} \psi_{+, a_1}^{\Omega_1} = \psi_{+, a_1}^{\Omega_0} + i\psi_{-, a_1}^{\Omega_0}, \\ \psi_{-, a_1}^{\Omega_1} = \psi_{-, a_1}^{\Omega_0}, \end{cases} \quad \begin{cases} \psi_{+, a_2}^{\Omega_0} = \psi_{+, a_2}^{\Omega_2} + i\psi_{-, a_2}^{\Omega_2}, \\ \psi_{-, a_2}^{\Omega_0} = \psi_{-, a_2}^{\Omega_2}, \end{cases}$$

because the sign of the Stokes curves are \oplus . It follows from (4.40) that

$$\begin{aligned} \psi_{\pm, a_1} &= \exp \pm \left(\int_{a_1}^{a_2} S_{\text{odd}} dx \right) \psi_{\pm, a_2} \\ &= e^{\pm \pi i c \eta} \psi_{\pm, a_2} \end{aligned}$$

hold. Furthermore, since $\psi_{\pm, \text{IM}}$ and ψ_{\pm, a_1} are related as

$$\psi_{\pm, \text{IM}} = e^{\pm(\frac{1}{2}U-V)} \psi_{\pm, a_1},$$

we can derive the connection formula for $\psi_{\pm, \text{IM}}$ by combining the above formulas:

$$\begin{cases} \psi_{+, \text{IM}}^{\Omega_1} = \psi_{+, \text{IM}}^{\Omega_2} + i(1 + e^{2\pi i c \eta})e^{U-2V} \psi_{-, \text{IM}}^{\Omega_2}, \\ \psi_{-, \text{IM}}^{\Omega_1} = \psi_{-, \text{IM}}^{\Omega_2}. \end{cases}$$

Thus we have

$$\mathfrak{s}_1 = i(1 + e^{2\pi i c \eta})e^{U-2V}. \quad (6.73)$$

• Computation of \mathfrak{s}_4 . Because we have to consider a connection problem on a Stokes curve emanating from the double turning point $x = \lambda_0$, we use the transformation theory prepared in Subsections 6.1 and 6.2. In view of (6.10) we find that the leading term of the transformation $\tilde{x} = x_0(x, t, c)$ maps the Stokes curve in question to either $i\mathbb{R}_{>0}$ or $i\mathbb{R}_{<0}$ in Figure 6.2, and it depends on the choice of the branch of $\Delta^{1/4}$. We now assume that the Stokes curve in question is mapped to $i\mathbb{R}_{>0}$. (In view of (3.3) and (3.4), the change of the choice of the branch of $\Delta^{1/4}$ is equivalent to the replacement $\alpha \mapsto -\alpha$.) Since the normalization of $\psi_{\pm, \text{IM}}$ in Figure 6.8 is the same as in Remark 6.5 and the substituted 1-parameter solution is normalized at $t = \infty$, we can use the relation (6.62) in this case. Using the connection formula (6.64) and (6.68) for $\tilde{\psi}_{\pm}$ on $i\mathbb{R}_{>0}$, we derive the following connection formula for $\psi_{\pm, \text{IM}}$:

$$\begin{cases} \psi_{+, \text{IM}}^{\Omega_4} = \psi_{+, \text{IM}}^{\Omega_5}, \\ \psi_{-, \text{IM}}^{\Omega_4} = \psi_{-, \text{IM}}^{\Omega_5} + m \psi_{+, \text{IM}}^{\Omega_5}, \end{cases} \quad (6.74)$$

where

$$m = (\tilde{\rho}(t_{\text{II}}) - 2\tilde{\sigma}(t_{\text{II}}))\sqrt{\frac{\pi}{2}}e^{-2\eta t_{\text{II}}}. \quad (6.75)$$

Moreover, since

$$\tilde{\rho}(t_{\text{II}}) - 2\tilde{\sigma}(t_{\text{II}}) = -2\sqrt{2}\alpha e^{2\eta t_{\text{II}}}$$

holds by (6.39), we have the required Stokes multiplier \mathfrak{s}_4 :

$$\mathfrak{s}_4 = m = -2\sqrt{\pi}\alpha. \quad (6.76)$$

Remark 6.8. All the Stokes multipliers \mathfrak{s}_j and \mathfrak{s}'_j ($1 \leq j \leq 6$) in (6.71) and (6.72) are independent of t . It is consistent with the theory of isomonodromic deformation.

Remark 6.9. The Stokes multipliers \mathfrak{s}_5 and \mathfrak{s}'_5 in (6.71) and (6.72) are equal to 0. The reason is as follows. In the computation of \mathfrak{s}_5 we use the connection formula (6.65), (6.69) for $\tilde{\psi}_{\pm}$, and hence we have

$$i(\tilde{\rho}(t_{\text{II}}) + 2\tilde{\sigma}(t_{\text{II}}))\sqrt{\frac{\pi}{2}}e^{2\eta t_{\text{II}}}$$

as the connection coefficient instead of (6.75). This quantity vanishes by (6.39).

6.4 Derivation of the connection formula for the parametric Stokes phenomena through the Stokes multipliers of (SL_{II})

Now we rederive the connection formulas describing the parametric Stokes phenomena for the 1-parameter solutions $\lambda_{\infty}(t, c, \eta; \alpha)$ and $\lambda_{\tau_1}(t, c, \eta; \alpha)$ of (P_{II}) by using the explicit form of the Stokes multipliers of (SL_{II}) computed in Subsection 6.3.

If a true solution represented by a 1-parameter solution $\lambda(t, c, \eta; \alpha)$ for $\arg c = \frac{\pi}{2} - \varepsilon$ and that by $\lambda(t, c, \eta; \tilde{\alpha})$ for $\arg c = \frac{\pi}{2} + \varepsilon$ coincide, then the corresponding Stokes multipliers $\mathfrak{s}_j(c; \alpha)$ and $\mathfrak{s}'_j(c; \tilde{\alpha})$ of (SL_{II}) should coincide, that is,

$$\mathcal{S}[\mathfrak{s}_j(c; \alpha)] = \mathcal{S}[\mathfrak{s}'_j(c; \tilde{\alpha})] \quad (1 \leq j \leq 6) \quad (6.77)$$

should hold. Hence, comparing $\mathfrak{s}_j(c; \alpha)$ and $\mathfrak{s}'_j(c; \tilde{\alpha})$ given by (6.71) and using Corollary 5.1, we find that

$$\mathcal{S}[\mathfrak{s}_j(c; \alpha)] = \mathcal{S}[\mathfrak{s}'_j(c; \tilde{\alpha})] \Rightarrow \tilde{\alpha} = \alpha \quad (6.78)$$

holds in the case where the 1-parameter solution $\lambda_\infty(t, c, \eta; \alpha)$ is substituted into the coefficients of (SL_{II}) and (D_{II}) . This result (6.78) is consistent with (3.26), that is, the parametric Stokes phenomenon does not occur to $\lambda_\infty(t, c, \eta; \alpha)$. Similarly, in the case of $\lambda_{\tau_1}(t, c, \eta; \alpha)$ being substituted, the comparison of $\mathfrak{s}_j(c; \alpha)$ and $\mathfrak{s}'_j(c; \tilde{\alpha})$ in (6.72) tells us that

$$\begin{aligned} \mathcal{S}[\mathfrak{s}_j(c; \alpha)] = \mathcal{S}[\mathfrak{s}'_j(c; \tilde{\alpha})] &\Rightarrow \alpha \mathcal{S}[e^W|_{\arg c = \frac{\pi}{2} - \varepsilon}] = \tilde{\alpha} \mathcal{S}[e^W|_{\arg c = \frac{\pi}{2} + \varepsilon}] \\ &\Rightarrow \tilde{\alpha} = (1 + e^{2\pi i c \eta}) \alpha, \end{aligned} \quad (6.79)$$

and this is consistent with (3.27). Thus, we have rederived the connection formulas for the parametric Stokes phenomena for 1-parameter solutions of (P_{II}) through the computation of the Stokes multipliers of (SL_{II}) .

A Homogeneity

The second Painlevé equation (P_{II}) with a large parameter is obtained by a change of variables

$$(w, z, a) := (\eta^{\frac{1}{3}} \lambda, \eta^{\frac{2}{3}} t, \eta c)$$

from the “original” Painlevé equation $\frac{d^2 w}{dz^2} = 2w^3 + zw + a$, that is, if $w(z, a)$ is a solution of the original Painlevé equation, then $\lambda(t, c, \eta)$ given by $\eta^{\frac{1}{3}} \lambda(t, c, \eta) = w(\eta^{\frac{2}{3}} t, \eta c)$ is a solution of (P_{II}) . Hence various quantities which appeared in this paper have a homogeneity with respect to the following scaling operation:

$$(x, t, c, \eta) \mapsto (r^{-\frac{1}{3}} x, r^{-\frac{2}{3}} t, r^{-1} c, r \eta) \quad (r > 0).$$

For example, the homogenous degree of λ_0 which is an algebraic function defined by $2\lambda_0^3 + t\lambda_0 + c = 0$ is $-\frac{1}{3}$, that is,

$$\lambda_0(r^{-\frac{2}{3}} t, r^{-1} c, r \eta) = \eta^{-\frac{1}{3}} \lambda_0(t, c).$$

We list the homogenous degrees of quantities below.

$$\lambda_k^{(0)}(t, c) : \left(k - \frac{1}{3}\right) \quad (k \geq 0), \quad (\text{A.1})$$

$$\nu_k^{(0)}(t, c) : \left(k - \frac{2}{3}\right) \quad (k \geq 0), \quad (\text{A.2})$$

$$\lambda^{(0)}(t, c, \eta) : -\frac{1}{3}, \quad (\text{A.3})$$

$$\nu^{(0)}(t, c, \eta) : -\frac{2}{3}, \quad (\text{A.4})$$

$$\Delta(t, c) : -\frac{2}{3}, \quad (\text{A.5})$$

$$\tau_j(c) : -\frac{2}{3} \quad (j = 1, 2, 3), \quad (\text{A.6})$$

$$R_k(t, c) : \left(k + \frac{2}{3}\right) \quad (k \geq 0), \quad (\text{A.7})$$

$$R(t, c, \eta) : +\frac{2}{3}, \quad (\text{A.8})$$

$$R_{\text{odd}}(t, c, \eta) : +\frac{2}{3}, \quad (\text{A.9})$$

$$\phi_{\text{II}}(t, c) : -1, \quad (\text{A.10})$$

$$\lambda_\ell^{(k)}(t, c) : \left(-\frac{1}{3} + \frac{1}{2}k + \ell\right) \quad (k \geq 0, \ell \geq 0), \quad (\text{A.11})$$

$$\lambda(t, c, \eta; \alpha) : -\frac{1}{3}, \quad (\text{A.12})$$

$$\nu(t, c, \eta; \alpha) := \eta^{-1} \frac{d}{dt} \lambda(t, c, \eta; \alpha) : -\frac{2}{3}, \quad (\text{A.13})$$

$$W(c, \eta) : 0, \quad (\text{A.14})$$

$$Q_{\text{II}}(x, t, c, \eta; \alpha) : -\frac{4}{3}, \quad (\text{A.15})$$

$$A_{\text{II}}(x, t, c, \eta; \alpha) : +\frac{1}{3}, \quad (\text{A.16})$$

$$S(x, t, c, \eta; \alpha) : +\frac{1}{3}, \quad (\text{A.17})$$

$$S_{\text{odd}}(x, t, c, \eta; \alpha) : +\frac{1}{3}, \quad (\text{A.18})$$

$$a_j(t, c) : -\frac{1}{3} \quad (j = 1, 2), \quad (\text{A.19})$$

$$\psi_{\pm, \infty}(x, t, c, \eta; \alpha) : -\frac{1}{6}, \quad (\text{A.20})$$

$$\psi_{\pm, \text{IM}}(x, t, c, \eta; \alpha) : -\frac{1}{6}, \quad (\text{A.21})$$

$$U(t, c, \eta; \alpha) : 0, \quad (\text{A.22})$$

$$V(t, c, \eta; \alpha) : 0. \quad (\text{A.23})$$

These facts can be easily verified by straightforward computations.

Proposition A.1. *The formal series below have the following homogenous degrees:*

$$x_{\text{II}}(x, t, c, \eta; \alpha) : -\frac{1}{2}, \quad (\text{A.24})$$

$$\sigma(t, c, \eta; \alpha) : 0, \quad (\text{A.25})$$

$$\rho(t, c, \eta; \alpha) : 0, \quad (\text{A.26})$$

$$t_{\text{II}}(t, c, \eta; \alpha) : -1. \quad (\text{A.27})$$

Proof. First we check the homogeneity of x_{II} , σ and ρ . Due to (6.10) we know that the homogenous degrees of x_0 , σ and ρ is $-\frac{1}{2}$, 0 and 0 respectively. In what follows we show that the homogenous degrees of $x_\ell^{(k)}$, $\sigma_\ell^{(k)}$ and $\rho_\ell^{(k)}$ are

$$x_\ell^{(k)} : \left(-\frac{1}{2} + \frac{1}{2}k + \ell\right) \quad (k \geq 0, \ell \geq 0), \quad (\text{A.28})$$

$$\sigma_\ell^{(k)} : \left(\frac{1}{2}k + \ell\right) \quad (k \geq 0, \ell \geq 0) \quad (\text{A.29})$$

$$\rho_\ell^{(k)} : \left(\frac{1}{2}k + \ell\right) \quad (k \geq 0, \ell \geq 0) \quad (\text{A.30})$$

by induction. We note that, since the claims (A.28) \sim (A.30) are true for $k \geq 1, \ell = 0$ by (6.5), it suffices to confirm (A.28) \sim (A.30) for $k = k', \ell = \ell'$ ($k', \ell' \geq 0$) under the assumptions that they are true for $0 \leq k \leq k' - 1, \ell \geq 0$ and $k = k', 0 \leq \ell \leq \ell' - 1$. In view of (6.6) the differential equation which determines $x_{\ell'}^{(k')}$ has the form

$$8x_0 \frac{\partial x_0}{\partial x} \left(x_0 \frac{\partial x_{\ell'}^{(k')}}{\partial x} + x_{\ell'}^{(k')} \frac{\partial x_0}{\partial x} \right) = r_{\ell'}^{(k')}(x, t, c), \quad (\text{A.31})$$

and the homogenous degree of $r_{\ell'}^{(k')}$ is $(-\frac{4}{3} + \frac{1}{2}k' + \ell')$ by induction hypothesis. (We note that $r_{\ell'}^{(k')}$ ($k', \ell' \geq 0$) are holomorphic in x and have a zero of order at least 1 at $x = \lambda_0$ because σ , ρ and E are determined by the equations (6.7) \sim (6.9).) Since $x_{\ell'}^{(k')}$ is given, as a unique holomorphic solution of (A.31) at $x = \lambda_0$, by

$$x_{\ell'}^{(k')}(x, t, c) = \frac{1}{x_0} \int_{\lambda_0}^x \frac{r_{\ell'}^{(k')}}{8x_0 \frac{\partial x_0}{\partial x}} dx, \quad (\text{A.32})$$

it is homogenous with degree $(-\frac{1}{2} + \frac{1}{2}k' + \ell')$. The homogenous degrees of $\sigma_{\ell'}^{(k')}$ and $\rho_{\ell'}^{(k')}$ can be computed from (6.7), (6.8) and (A.28). Furthermore, taking into account that the homogenous degree of t_0 defined by (6.32) is -1 and that the formal power series $t_{\text{II}}^{(k)}$ ($k \geq 0$) are defined by (6.37) and (6.38), we find that the homogenous degree of the formal series t_{II} is -1 . \square

B Holomorphy of Z at $c = 0$

$Z = Z(c, \eta)$ introduced in Section 6 is a formal power series given by

$$Z = \frac{1}{2}U^{(0)} + \int_{\infty}^x (S_{\text{odd}}^{(0)} - \eta S_{-1}) dx - \eta(t_{\text{II}}^{(0)} - t_0) - \eta(x_{\text{II}}^{(0)2} - x_0^2). \quad (\text{B.1})$$

The right-hand side of (B.1) is independent of both x and t . (We can verify this fact by using (6.41).) The aim of Appendix B is to show that the coefficients $Z_{\ell}(c)$ of $\eta^{-\ell}$ in $Z(c, \eta)$ are holomorphic at $c = 0$ (for all $\ell \geq 0$) when the 1-parameter solution λ_{∞} normalized at $t = \infty$ is substituted into the coefficients of (SL_{II}) and (D_{II}) .

Let t_* be a point fixed in the domain in Figure 2.3, and $\delta_1, \delta_2, \delta_3$ be positive numbers satisfying that

$$|c| < \delta_3 \Rightarrow \{t; |t - t_*| \leq \delta_2\} \not\subset \tau_1(c), \tau_2(c), \tau_3(c), \quad (\text{B.2})$$

$$|c| < \delta_3, |t - t_*| < \delta_2 \Rightarrow \{x; |x - \lambda_0(t, c)| \leq \delta_1\} \not\subset a_1(t, c), a_2(t, c). \quad (\text{B.3})$$

Note that, since all the three P -turning points $\tau_j(c)$ of (P_{II}) tend to $t = 0$ in the t -plane and the two simple turning points $x = a_j$ of (SL_{II}) tend to $x = -\lambda_0(t, 0) = +\frac{i}{\sqrt{2}}t^{1/2}$ in the x -plane when c tends to 0, we can take such positive numbers δ_j . For the above $\delta_1, \delta_2, \delta_3$ and t_* , we define domains $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ and \mathcal{D} by

$$\mathcal{D}_3 := \{c; |c| < \delta_3\},$$

$$\mathcal{D}_2 := \{t; |t - t_*| < \delta_2\},$$

$$\mathcal{D}_1(t, c) := \{x; |x - \lambda_0(t, c)| < \delta_1\},$$

$$\mathcal{D} := \{(x, t, c); (t, c) \in \mathcal{D}_2 \times \mathcal{D}_3, x \in \mathcal{D}_1(t, c)\}.$$

We will verify that all the coefficients of the formal power series which appear in the right-hand side of (B.1) such as $U^{(0)}$, $t_{\text{II}}^{(0)}$, $x_{\text{II}}^{(0)}$, etc., are holomorphic on \mathcal{D} . ($\delta_1, \delta_2, \delta_3$ may be chosen sufficiently small again if necessary.)

Lemma B.1. *All the coefficients of the formal power series*

$$U^{(0)}(t, c, \eta) = \int_{\infty}^t (\lambda^{(0)}(t, c, \eta) - \lambda_0(t, c)) dt$$

are holomorphic on $\mathcal{D}_2 \times \mathcal{D}_3$.

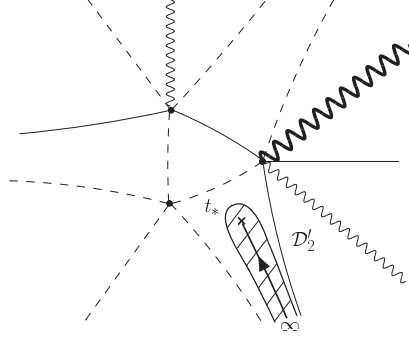


Figure B.1: Domain \mathcal{D}'_2 .

Proof. Let \mathcal{D}'_2 be a domain in t -plane given by

$$\mathcal{D}'_2 = \bigcup_{t'} \{t; |t - t'| < \delta_2\} \quad (\supset \mathcal{D}_2).$$

(Here t' runs over all points on the path of integration of $U^{(0)}(t_*, c, \eta)$. See Figure 3.2 for the choice of the path of integration.) We show that all the coefficients of $\lambda^{(0)} - \lambda_0$ are holomorphic on $\mathcal{D}'_2 \times \mathcal{D}_3$ and integrable at $t = \infty$. (δ_2 and δ_3 should be chosen smaller so that

$$|c| < \delta_3 \Rightarrow \overline{\mathcal{D}'_2} \not\ni \tau_1(c), \tau_2(c), \tau_3(c)$$

holds since the coefficients of $\lambda^{(0)}$ are singular at P -turning points.)

Since the discriminant of the algebraic equation $2\lambda^3 + t\lambda + c = 0$ for λ never vanishes on $\mathcal{D}'_2 \times \mathcal{D}_3$, λ_0 is holomorphic on $\mathcal{D}'_2 \times \mathcal{D}_3$ and

$$\Delta = 6\lambda_0^2 + t = \frac{4\lambda_0^3 - c}{\lambda_0}$$

is also holomorphic and never vanishes on $\mathcal{D}'_2 \times \mathcal{D}_3$. The holomorphy of all the coefficients on $\mathcal{D}'_2 \times \mathcal{D}_3$ follows from these facts and the recursive relations (2.3). Indeed, by induction, we can confirm that the coefficient $\lambda_k^{(0)}$ is identically 0 when k is an odd number and has the form

$$\lambda_{2n}^{(0)}(t, c) = \frac{p_{2n}(\lambda_0, c)}{(4\lambda_0^3 - c)^{5n-1}} \quad (\text{B.4})$$

when $k = 2n$ ($n \geq 1$) is an even number, where $p_{2n} \in \mathbb{C}[\lambda_0, c]$ is a polynomial with $\deg_{\lambda_0}(p_{2n}) \leq 9n - 2$. (Here $\deg_{\lambda_0}(p_{2n})$ is the degree of the polynomial p_{2n} when it is considered a polynomial of λ_0 .) Thus the holomorphy of $\lambda_k^{(0)}$ is obvious. In view of (B.4) and

$$\frac{\partial}{\partial c} \lambda_0 = -\frac{1}{\Delta},$$

the c -derivative of $\lambda_{2n}^{(0)}$ is also represented as

$$\frac{\partial}{\partial c} \lambda_{2n}^{(0)}(t, c) = \frac{\hat{p}_{2n}(\lambda_0, c)}{(4\lambda_0^3 - c)^{5n+1}} \quad (n \geq 1), \quad (\text{B.5})$$

where $\hat{p}_{2n}(\lambda_0, c) \in \mathbb{C}[\lambda_0, c]$ is also a polynomial with $\deg_{\lambda_0}(\hat{p}_{2n}) \leq 9n + 4$. Since the behavior of λ_0 when $t \rightarrow \infty$ is given by (2.17), (B.5) implies that $\frac{\partial}{\partial c} \lambda_{2n}^{(0)}(t, c)$ is integrable at $t = \infty$ uniformly with respect to $c \in \mathcal{D}_3$. Therefore,

$$\int_{\infty}^t \lambda_{2n}^{(0)}(t, c) dt$$

is holomorphic on $\mathcal{D}'_2 \times \mathcal{D}_3 (\supset \mathcal{D}_2 \times \mathcal{D}_3)$ for all $n \geq 1$. □

Lemma B.2. *For the 1-parameter solution*

$$\begin{cases} \lambda_\infty(t, c, \eta; \alpha) &= \lambda^{(0)}(t, c, \eta) + \alpha\eta^{-\frac{1}{2}}\lambda_\infty^{(1)}(t, c, \eta)e^{\eta\phi_\Pi} + (\alpha\eta^{-\frac{1}{2}})^2\lambda_\infty^{(2)}(t, c, \eta)e^{2\eta\phi_\Pi} + \dots \\ \nu_\infty(t, c, \eta; \alpha) &= \eta^{-1}\frac{d}{dt}\lambda_\infty(t, c, \eta; \alpha) \\ &= \nu^{(0)}(t, c, \eta) + \alpha\eta^{-\frac{1}{2}}\nu_\infty^{(1)}(t, c, \eta)e^{\eta\phi_\Pi} + (\alpha\eta^{-\frac{1}{2}})^2\nu_\infty^{(2)}(t, c, \eta)e^{2\eta\phi_\Pi} + \dots \end{cases}$$

of (H_Π) normalized at $t = \infty$, all the coefficients of the formal power series $\lambda_\infty^{(k)}(t, c, \eta)$, $\nu_\infty^{(k)}(t, c, \eta)$ ($k \geq 0$) are holomorphic on $\mathcal{D}_2 \times \mathcal{D}_3$.

Proof. The holomorphy of all the coefficients of $\lambda^{(0)}$ has been already shown in the proof of Lemma B.1. Next we consider

$$\lambda_\infty^{(1)} = \frac{1}{\sqrt{\eta^{-1}R_{\text{odd}}}} \exp\left(\int_\infty^t (R_{\text{odd}} - \eta R_{-1}) dt\right).$$

It is clear that $R_{-1}(t, c) = \sqrt{\Delta(t, c)}$ is holomorphic on $\mathcal{D}_2' \times \mathcal{D}_3$. Due to the recursive relations (2.9) for $R_k(t, c)$, we can obtain the following expression by induction:

$$R_{2n}(t, c) = \frac{q_{2n}(\lambda_0, c)}{(4\lambda_0^3 - c)^{5n+2}} \quad (n \geq 0), \quad (\text{B.6})$$

$$R_{2n+1}(t, c) = \frac{\lambda_0^{\frac{1}{2}} q_{2n+1}(\lambda_0, c)}{(4\lambda_0^3 - c)^{5n+\frac{9}{2}}} \quad (n \geq 0), \quad (\text{B.7})$$

where $q_{2n}(\lambda_0, c), q_{2n+1}(\lambda_0, c) \in \mathbb{C}[\lambda_0, c]$ are polynomials satisfying $\deg_{\lambda_0}(q_{2n}) \leq 9n+4$ and $\deg_{\lambda_0}(q_{2n+1}) \leq 9n+8$. Similarly to the proof of Lemma B.1, we can show that

$$\int_\infty^t R_{2n+1}(t, c) dt$$

is holomorphic on $\mathcal{D}_2' \times \mathcal{D}_3 (\supset \mathcal{D}_2 \times \mathcal{D}_3)$ by using the expression (B.7) for all $n \geq 0$. Thus the holomorphy of the coefficients of $\lambda_\infty^{(1)}$ is verified. The holomorphy of the coefficients of $\lambda_\infty^{(k)}$ ($k \geq 2$) can be confirmed from the recursive relations (2.14), and the holomorphy of the coefficients of $\nu^{(0)}$ and $\nu_\infty^{(k)}$ ($k \geq 1$) can also be shown by using the relation $\nu_\infty(t, c, \eta; \alpha) = \eta^{-1} \frac{d}{dt} \lambda_\infty(t, c, \eta; \alpha)$. \square

Remark B.1. When c tends to 0, the three P -turning points merge to $t = 0$ simultaneously. Hence, if we consider a 1-parameter solution normalized at a P -turning point $t = \tau_1$, we can not expect that Lemma B.2 holds because the integration path of $\int_{\tau_1}^t R_{\text{odd}} dt$ is pinched by the turning points and all the coefficients of the formal power series R_{odd} have a singularity at P -turning points. The assumption that a 1-parameter solution is normalized at $t = \infty$ is essential.

It follows from Lemma B.2 that, if λ_∞ is substituted into Q_Π , then all the coefficients $Q_\ell^{(k)}$ are holomorphic on $\mathcal{D} \setminus \{x = \lambda_0\}$ and bounded as x tends to ∞ when $(k, \ell) \neq (0, 0)$. (Note that $Q_0^{(0)} = Q_0 = (x - \lambda_0)^2(x^2 + 2\lambda_0x + 3\lambda_0^2 + t)$ for $(k, \ell) = (0, 0)$.) More precisely, $Q_\ell^{(k)}$ are represented as

$$Q_\ell^{(k)}(x, t, c) = \frac{u(x, t, c)}{(x - \lambda_0)^m} \quad (\text{B.8})$$

for some integer $m \geq 0$ and a polynomial $u(x, t, c)$ of x with coefficients being holomorphic on $\mathcal{D}_2 \times \mathcal{D}_3$ and $\deg_x(u) \leq m$.

Lemma B.3. Assume that the 1-parameter solution λ_∞ normalized at $t = \infty$ is substituted into Q_{II} . Then all the coefficients of the formal power series

$$\int_{\infty}^x (S_{\text{odd}}^{(0)}(x, t, c, \eta) - \eta S_{-1}(x, t, c)) dx$$

are holomorphic on \mathcal{D} and all the coefficients of

$$\int_{\infty}^x S_{\text{odd}}^{(k)}(x, t, c, \eta) dx$$

are holomorphic on $\mathcal{D} \setminus \{x = \lambda_0\}$. Here the paths of integration of the above integrals are taken as in Figure 6.1.

Proof. For $(t, c) \in \mathcal{D}_2 \times \mathcal{D}_3$ let $\mathcal{D}'_1(t, c)$ be a domain in the x -plane defined by

$$\mathcal{D}'_1(t, c) = \bigcup_{x'} \{x; |x - x'| < \delta_1\} \quad (\supset \mathcal{D}_1(t, c)),$$

where x' runs over all points on the path of integration of $\int_{\infty}^{\lambda_0} (S_{\text{odd}}^{(0)} - \eta S_{-1}) dx$, and let \mathcal{D}' be the following domain:

$$\mathcal{D}' := \{(x, t, c); (t, c) \in \mathcal{D}_2 \times \mathcal{D}_3, x \in \mathcal{D}'_1(t, c)\} \quad (\supset \mathcal{D}).$$

We prove the holomorphy and integrability at $x = \infty$ of all the coefficients of $S_{\text{odd}}^{(0)} - \eta S_{-1}$

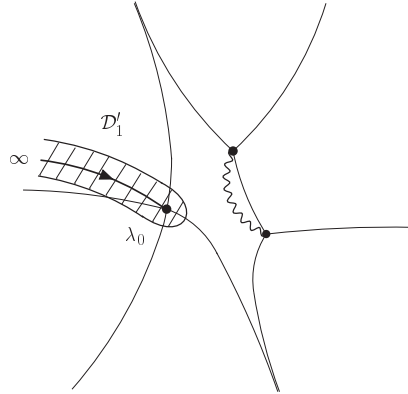


Figure B.2: Domain \mathcal{D}'_1 .

and $S_{\text{odd}}^{(k)}$. ($\delta_1, \delta_2, \delta_3$ should be chosen smaller so that

$$(t, c) \in \mathcal{D}_2 \times \mathcal{D}_3 \Rightarrow \overline{\mathcal{D}'_1(t, c)} \not\ni a_1(t, c), a_2(t, c)$$

holds, if necessary, since the coefficients of $S_{\text{odd}}^{(k)}$ are singular at turning points. The above condition guarantees that $x^2 + 2\lambda_0 x + 3\lambda_0^2 + t \neq 0$ on \mathcal{D}' .)

By the recursive relations (4.18) and (B.8), for $(k, \ell) \neq (0, 0)$, we obtain the following expression of $S_{\text{odd}, \ell}^{(k)}$:

$$S_{\text{odd}, \ell}^{(k)}(x, t, c) = \frac{v(x, t, c)}{(x - \lambda_0)^{n_1} (x^2 + 2\lambda_0 x + 2\lambda_0^2 + t)^{\frac{n_2}{2}}}, \quad (\text{B.9})$$

where $n_1, n_2 \geq 0$ are some integers and $v(x, t, c)$ is a polynomial of x whose coefficients are all holomorphic on $\mathcal{D}_2 \times \mathcal{D}_3$ and which satisfies $\deg_x(v) \leq n_1 + n_2 - 2$. Due to (B.9), we have

$$\frac{\partial}{\partial t} S_{\text{odd}, \ell}^{(k)}(x, t, c) = \frac{\tilde{v}(x, t, c)}{(x - \lambda_0)^{n_1+1} (x^2 + 2\lambda_0 x + 2\lambda_0^2 + t)^{\frac{n_2}{2}+1}}, \quad (\text{B.10})$$

where $\tilde{v}(x, t, c)$ is also a polynomial of x whose coefficients are all holomorphic on $\mathcal{D}_2 \times \mathcal{D}_3$ and which satisfies $\deg_x(\tilde{v}) \leq n_1 + n_2 + 1$. The c -derivative of $S_{\text{odd}, \ell}^{(k)}$ also has a form similar to (B.10). These facts imply that $\frac{\partial}{\partial t} S_{\text{odd}, \ell}^{(k)}$ and $\frac{\partial}{\partial c} S_{\text{odd}, \ell}^{(k)}$ are both integrable at $x = \infty$ uniformly with respect to $(t, c) \in \mathcal{D}_2 \times \mathcal{D}_3$. Therefore all the coefficients of $\int_{\infty}^x (S_{\text{odd}}^{(0)} - \eta S_{-1}) dx$ and $\int_{\infty}^x S_{\text{odd}}^{(k)} dx$ ($k \geq 1$) are holomorphic on $\mathcal{D}' \setminus \{x = \lambda_0\}$ ($\supset \mathcal{D} \setminus \{x = \lambda_0\}$). Furthermore, as noted in Lemma 6.1(ii), all the coefficients of $S_{\text{odd}}^{(0)}$ are holomorphic at $x = \lambda_0$. Thus we have the holomorphy on \mathcal{D}' ($\supset \mathcal{D}$) of the coefficients of $\int_{\infty}^x (S_{\text{odd}}^{(0)} - \eta S_{-1}) dx$. \square

Next we discuss the holomorphy of the coefficients of $x_{\text{II}}^{(k)}$ and $t_{\text{II}}^{(k)}$ on \mathcal{D} .

Lemma B.4. *Assume that the 1-parameter solution λ_{∞} normalized at $t = \infty$ is substituted into the coefficients of (SL_{II}) and (D_{II}) . Then all the coefficients of the formal power series*

$$x_{\text{II}}^{(k)}(x, t, c, \eta) \quad (k \geq 0)$$

are holomorphic on \mathcal{D} .

Proof. We prove this lemma by induction. Due to the condition (B.3), $x_0(x, t, c) = \left[\int_{\lambda_0}^x \sqrt{Q_0(x, t, c)} dx \right]^{\frac{1}{2}}$ is holomorphic on \mathcal{D} . The x -derivative $\frac{\partial x_0}{\partial x}$ of x_0 behaves as

$$\frac{\partial x_0}{\partial x} = \frac{1}{\sqrt{2}} \Delta^{\frac{1}{4}} + \mathcal{O}(x - \lambda_0)$$

when x tends to λ_0 . Hence, since $\Delta \neq 0$ on $\mathcal{D}_2 \times \mathcal{D}_3$, we can assume that $\frac{\partial x_0}{\partial x} \neq 0$ holds on \mathcal{D} (by taking sufficiently small $\delta_1 > 0$).

Since the claims are true for $k \geq 1, \ell = 0$ by (6.5), it suffices to confirm the holomorphies of $x_{\ell'}^{(k')}$ ($k', \ell' \geq 0$) under the assumptions that $x_{\ell}^{(k)}$ are holomorphic on \mathcal{D} for $0 \leq k \leq k' - 1, \ell \geq 0$ and $k = k', 0 \leq \ell \leq \ell' - 1$. As noted in the proof of Proposition A.1, the differential equation which determines $x_{\ell'}^{(k')}$ is given by (A.31) and $r_{\ell'}^{(k')}$ is holomorphic on $\mathcal{D} \setminus \{x = \lambda_0\}$ under the assumptions of induction. (Note that $1/x_0$ and $Q_{\ell}^{(k)}$ have a singularity at $x = \lambda_0$.) However, it follows from the proof of [AKT1, Theorem 3.1] that $r_{\ell'}^{(k')}$ is holomorphic and has a zero of order 1 at $x = \lambda_0$ because σ, ρ and E satisfy (6.7) \sim (6.9). Thus $x_{\ell'}^{(k')}$ given by (A.32) is holomorphic on \mathcal{D} . \square

Lemma B.5. *Assume that the 1-parameter solution λ_{∞} normalized at $t = \infty$ is substituted into the coefficients of (SL_{II}) and (D_{II}) . Then all the coefficients of the formal power series*

$$\begin{aligned} t_{\text{II}}^{(0)}(t, c, \eta) - t_0(t, c), \\ t_{\text{II}}^{(k)}(t, c, \eta) \quad (k \geq 1), \end{aligned}$$

are holomorphic on $\mathcal{D}_2 \times \mathcal{D}_3$.

Proof. By Lemma B.4,

$$\frac{\partial^n x_{\ell}^{(k)}}{\partial x^n}(\lambda_0, t, c)$$

is holomorphic on $\mathcal{D}_2 \times \mathcal{D}_3$ for $k, \ell, n \geq 0$. Moreover, since

$$\frac{\partial x_0}{\partial x}(\lambda_0, t, c) \neq 0$$

holds on $\mathcal{D}_2 \times \mathcal{D}_3$, all the coefficients of $\sigma^{(k)}(t, c, \eta)$ ($k \geq 0$) are holomorphic on $\mathcal{D}_2 \times \mathcal{D}_3$ by the definition (6.7) of σ . Therefore, the holomorphy of all the coefficients of $t_{\text{II}}^{(0)}(t, c, \eta) - t_0(t, c)$ and $t_{\text{II}}^{(k)}(t, c, \eta)$ ($k \geq 1$) immediately follows from (6.37) and (6.38). \square

Thus we finally obtain the following:

Proposition B.1. *Assume that the 1-parameter solution λ_∞ normalized at $t = \infty$ is substituted into the coefficients of (SL_Π) and (D_Π) . Then all the coefficients of the formal power series $Z(c, \eta)$ which is given by (B.1) are holomorphic at $c = 0$.*

Proof. By Lemmas B.1 \sim B.5, we have shown that all the coefficients of Z are holomorphic on \mathcal{D} if λ_∞ is substituted. Moreover, since Z is independent of x and t , they are holomorphic functions of c on \mathcal{D}_3 , which is a neighborhood of $c = 0$. \square

References

- [AKT1] T.Aoki, T.Kawai and Y.Takei : WKB analysis of Painlevé transcendents with a large parameter.II. — Multiple-scale analysis of Painlevé transcendents, Structure of Solutions of Differential Equations, World Scientific, 1996, pp.1-49.
- [AKT2] T.Aoki, T.Kawai and Y.Takei : The Bender-Wu analysis and the Voros theory.II, Advanced Studies in Pure Mathematics, **54**, Math. Soc. Japan, 2009, pp.19-94.
- [C] O.Costin : Asymptotics and Borel Summability, Monographs and Surveys in Pure and Applied Mathematics 141, Chapman & Hall/CRC, New York, 2009.
- [DP] E.Delabaere and F.Pharm : Resurgent methods in semi-classical asymptotics, Ann. Inst. Henri Poincaré , **71**(1999), 1-94.
- [JM] M.Jimbo and T.Miwa : Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II , Physica, **2D**(1981), 407-448.
- [KT1] T.Kawai and Y.Takei : WKB analysis and deformation of Schrödinger equations, RIMS Kôkyûroku, No. **854**, 1993, pp.22-42.
- [KT2] T.Kawai and Y.Takei : WKB analysis of Painlevé transcendents with a large parameter.I, Adv.in Math., **118**(1996),1-33.
- [KT3] T.Kawai and Y.Takei : WKB analysis of Painlevé transcendents with a large parameter.III, Adv.in Math., **134**(1998), 178-218.
- [KT4] T.Kawai and Y.Takei : Algebraic Analysis of Singular Perturbation Theory, Translations of Mathematical Monographs, volume 227, American Mathematical Society, 2005.
- [KT5] T.Kawai and Y.Takei : WKB analysis of higher order Painlevé equations with a large parameter — Local reduction of 0-parameter solutions for Painlevé hierarchies (P_J) ($J=I, II-1$ or $II-2$), Adv.in Math., **203**(2006), 636-672.
- [T1] Y.Takei : An explicit description of the connection formula for the first Painlevé equation, Toward the Exact WKB Analysis of Differential Equations, Linear or Non-Linear, Kyoto Univ. Press, 2000, pp.271-296.
- [T2] Y.Takei : Sato's conjecture for the Weber equation and transformation theory for Schrödinger equations with a merging pair of turning points, RIMS Kôkyûroku Bessatsu, **B10**(2008), 205-338.
- [V] A.Voros : The return of the quartic oscillator. The complex WKB method, Ann. Inst. Henri Poincaré, **39**(1983), 211-388.